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**Magdalena MARTINÁSKOVÁ<sup>1</sup>, Miroslav VOŘECHOVSKÝ<sup>2</sup>****FAILURE PROBABILITY ESTIMATION USING ASYMPTOTIC SAMPLING AND ITS  
DEPENDENCE UPON THE SELECTED SAMPLING SCHEME****Abstract**

The article examines the use of Asymptotic Sampling (AS) for the estimation of failure probability. The AS algorithm requires samples of multidimensional Gaussian random vectors, which may be obtained by many alternative means that influence the performance of the AS method. Several reliability problems (test functions) have been selected in order to test AS with various sampling schemes: (i) Monte Carlo designs; (ii) LHS designs optimized using the Periodic Audze-Eglājs (PAE) criterion; (iii) designs prepared using Sobol' sequences. All results are compared with the exact failure probability value.

**Keywords**

Asymptotic Sampling, failure probability, quasi-Monte Carlo, LHS designs.

**1 INTRODUCTION**

The main task of any civil engineer is to ensure a designed structure has a certain level of reliability, i.e. an appropriate failure probability ( $P_f$ ). Each design encompasses a large number of random input variables that have to be dealt with. Apart from the common practice of ensuring sufficient reliability by using partial safety factors, it is also possible to use the full probabilistic design. This approach is particularly advantageous in the case of the design of an extraordinary structure/structural system/material. It may also be beneficial when designing a frequently used structural member (e.g. railway sleepers, etc.). Production in high quantities has a potential for considerable financial savings if the design is economical.

In the case of the full probabilistic approach, a large variety of methods exist for  $P_f$  calculation (often limited to certain types of problems). Methods for  $P_f$  approximation have been developed as well. The most elementary method is to perform a certain number of simulations ( $N_{\text{sim}}$ ) and estimate  $P_f$  as a quotient obtained by dividing the number of failures by the total number of simulations,  $N_{\text{sim}}$  [1, 2]. The easiest way of selecting the simulations is via the Monte Carlo (MC) method (described later). This elementary means of  $P_f$  estimation is usually ineffective for civil engineering structures due to the large variance of the estimator. Since the target failure probability values are very small (in the order of  $10^{-6}$ ), crude MC sampling demands a very high number of simulations. As each simulation evaluation usually requires a long time, the amount of total time required becomes unacceptable. Therefore, approximation methods are exploited. These decrease the time necessary for calculation. The first possibility is to decrease the complexity of the problem – the difficulty of evaluating whether the simulation belongs to the failure domain or not. Methods based on the response surface [3] work with this conception. The other way is to decrease the necessary number of simulations for  $P_f$

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estimation. This approach is exploited by, e.g. Importance sampling [4], Subset simulation [5] or Asymptotic sampling (AS; [6]).

This article deals with the AS method, which allows the estimation of low  $P_f$  using rather low  $N_{\text{sim}}$  by artificially increasing the variance of input random variables. This causes an increase in  $P_f$  and, therefore, a decrease in the necessary number of simulations.

The AS procedure requires a computer experiment to be executed, thus raising the question as to whether the accuracy of AS is affected by the way the experiment is prepared. Three test functions were selected to test this hypothesis. Their description follows in the next section. The failure probability of each of these test functions was repeatedly estimated using AS. Afterwards, the estimates were statistically processed and compared with the exact value of  $P_f$  for the individual test functions.

## 2 TEST FUNCTIONS

The selected functions mimic a reliability margin which (in an engineering problem) is usually the difference between the resistance of the structure and the load effect. Elementary functions (further denoted by  $g$ ) were selected; therefore, it is easy to determine whether the simulation belongs to the failure domain (calculation of the test function using the simulation coordinates results in a negative value:  $g < 0$ ) or the safe domain ( $g > 0$ ).

The input random variables have normalized normal distribution (their distribution function is denoted by  $\Phi$  throughout the paper).

### 2.1 Sum1D

The first function is a summation of one input random variable and a constant, 4.75, which ensures the resulting  $P_f$  has a failure probability typical of standard civil engineering structures

$$g_1(x) = x + 4.75 \quad (1)$$

The boundary between the safe and failure (feasible and infeasible) domains is  $g_1 = 0$ , and the failure occurs if  $g_1 < 0$  ( $x < -4.75$ ); therefore, the exact reliability index value may be determined analytically ( $\beta = 4.75$ ) as well as the failure probability ( $P_f = \Phi(-4.75) \cong 1.0171 \cdot 10^{-6}$ ).

### 2.2 Sum2D

The next function is the sum of two input random variables with another constant, 6.7 (the design domain is a 2-dimensional space of input random variables).

$$g_2(x_1, x_2) = x_1 + x_2 + 6.7 \quad (2)$$

The coordinates of the design point are  $(-3.35, -3.35)$  and thus the exact value of the reliability index  $\beta = 3.35\sqrt{2} \cong 4.7376$ . The corresponding failure probability is  $P_f \cong 1.0812 \cdot 10^{-6}$ .

### 2.3 Sin2D

The last test function works with two input random variables and a third constant, 4, and it contains the sine function.

$$g_3(x_1, x_2) = -\frac{x_1}{4} + \sin(5x_1) - x_2 + 4 \quad (3)$$

This form of the test function causes a wavy boundary to arise between the safe and failure domains (see Fig. 1). Therefore, it is not possible to solve this problem analytically based on simple geometry (as in the previous two cases) nor by some of the most common and elementary approximation methods (FORM, SORM [1]), because such a boundary comprises many design points.

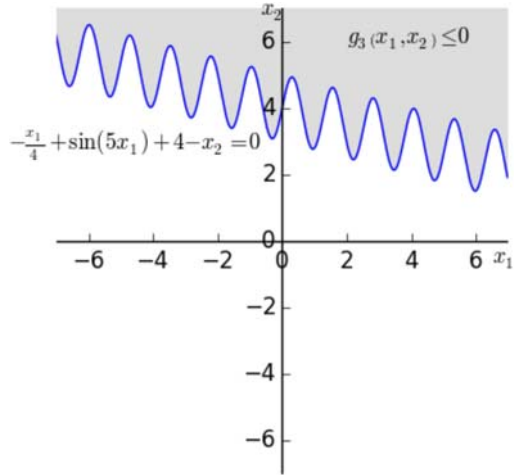


Fig.1: Failure domain of the function Sin2D

The “exact value” of  $P_f$  used for comparison cannot be obtained analytically in this case. The crude Monte Carlo method has been used instead. This method provides an unbiased  $P_f$  estimate which is accurate if a high number of simulations are used. In this case, the estimated failure probability is  $P_f \cong 4,1508 \cdot 10^{-4}$  ( $10^9$  simulations have been used). The corresponding reliability index  $\beta \cong 3.3425$ .

### 3 ASYMPTOTIC SAMPLING (AS)

AS is one of the approximation methods that lower the time requirement of  $P_f$  estimation by decreasing the number of necessary simulations ( $N_{sim}$ ). It works for problems with a very low failure probability and high number of random input variables ( $N_{var}$ ) as well.

$P_f$  is replaced by the Cornell reliability index  $\beta$  in all the calculations. The relation between these two assumes normality of the safety margin and therefore may easily be expressed as

$$\beta = -\Phi^{-1}(P_f) \quad (4)$$

The principle of the AS method is to artificially increase the variance of all the random input variables in subsequent steps. This causes the  $P_f$  to increase as well, and it is possible to estimate it with lower  $N_{sim}$  in each step (see Fig. 2). This makes it easy to obtain the  $P_f$  estimate for artificially created problems with higher variance of random input variables and then extrapolate the failure probability for the original problem afterwards.

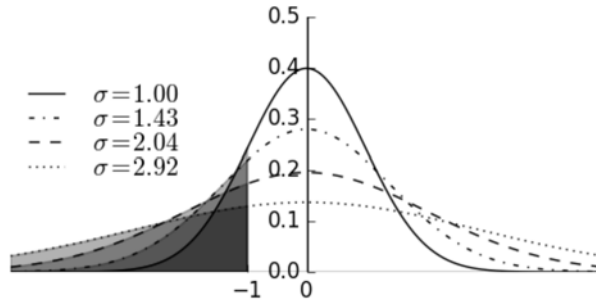


Fig.2: Artificial increase in variance in subsequent steps of AS in the case of a 1D function for which the design point is in -1 (the  $P_f$  in the subsequent steps is marked by the hatched region)

A detailed description of the procedure may be found in [6, 7], for example.

## 4 SAMPLING METHODS

To evaluate the  $P_t$  in individual AS steps, it is necessary to perform a computer experiment. Therefore, the position of samples (simulation points) in the design space has to be prepared in advance (design of experiment).

The sample coordinates for the computer experiment have been generated by (i) the Monte Carlo method (MC); (ii) Latin Hypercube Sampling (LHS), while the placement of the samples has further been optimized using the Periodic Audze-Eglājs (PAE) criterion and, (iii) the quasi-Monte Carlo method (QMC), particularly Sobol' sequence (see Fig. 3). These methods are briefly described below.

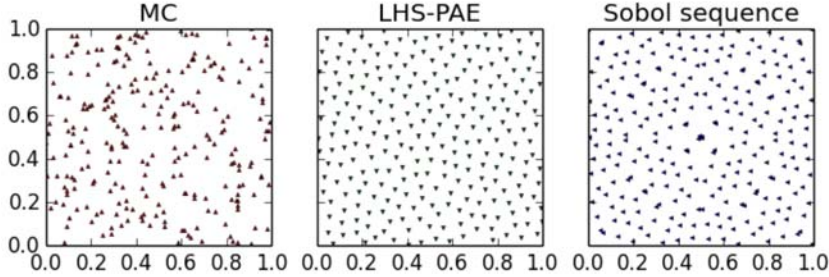


Fig.3: The positions of samples in the design space produced by the crude MC method, LHS optimized by the PAE criterion and QMC design exploiting Sobol' sequences;  $N_{var} = 2$ ,  $N_{sim} = 256$

### 4.1 Monte Carlo (MC)

This method consists in the random selection of coordinates with respect to the probability distribution of all the random input variables. It should result in unbiased estimates. However, these estimates may have rather large variance. The consequences for the AS results are described later.

### 4.2 Latin Hypercube Sampling (LHS)

LHS is a special type of MC method. It is a stratified simulation method that (unlike crude MC) ensures more uniform filling of the design space with respect to probability, which (in certain types of problems) enables the use of lower  $N_{sim}$  to obtain a resulting estimate of similar accuracy and lower variance than MC.

Even more uniform filling of the design space and lower resulting variance may be attained via the optimization of such designs. This has been exploited in this study as well. The criterion used for optimization is Periodic Audze-Eglājs, which is thoroughly described in [8, 9].

### 4.3 Quasi-Monte Carlo (QMC) designs – Sobol' sequences

QMC designs are in fact deterministic designs (the sampling point coordinates are defined by a function that exclusively depends on the type of sequence used). Nevertheless, they retain (at least partly) the advantages of stochastic designs, especially their non-collapsibility.

A great variety of low-discrepancy sequences exist for the creation of QMC designs; e.g. the van der Corput sequence [10], the Halton sequence [11] and the Niederreiter sequence [12]. This article exploits the Sobol' sequence, which is described in [13], for example.

## 5 RESULTS

This section contains graphs showing the results of individual AS steps, regression curves and final  $\beta$  estimates. In the case of stochastic designs (MC, LHS), the analysis has been performed repeatedly (10 times) and the results have been statistically processed. The graphs, therefore, show not only  $\beta$  estimates but also the variance of these estimates. The quasi-random designs prepared by Sobol' sequences are always identical for a given ( $N_{sim}$ ,  $N_{var}$ ) combination, thus repeated evaluation is not carried out. Each of the graphs shows the exact  $\beta$  value according to Section 2, as well.

Apart from that, the graphs display an auxiliary axis to present corresponding  $P_t$  values.

The AS method requires that certain parameters be set in advance. In this work, the parameters have been selected (the designation of all the parameters is identical to that found in the work mentioned above [7]) as follows:

- $f = 1$  the initial value of  $f$  (the first step of AS is a real problem without the adjustment of the variance),
- $f_{\text{fact}} = 0.95$  the coefficient of  $f(1/(f \cdot f_{\text{fact}}^i))$  is the coefficient for the variance increase in the  $i$ -th step),
- $N_{\text{sim,step}} = 256$  the number of simulations in each step of AS (number of sampling points in design plan of computer experiment in each of the steps of AS),
- $N_{f,\text{step}} = 7$  the minimum number of failures in a step (if the step finishes with a lower number of failures, it is considered invalid and a further increase in the variance of random input variables is necessary),
- $K_{\text{req}} = 5$  the minimum number of  $f$  and  $\beta$  pairs for performing regression (if this number is not reached, more AS steps are necessary),
- $\beta = A \cdot f + B/f$  the regression function used for the  $P_f$  extrapolation in the original problem.

Tab.1: The estimates of the  $\beta$

Test function (values of $\beta$ and $P_f$ according to section 2)	Sampling method	Average $\beta$ (corresponding $P_f$ )	Sample standard deviation of $\beta$
Sum1D $\beta = 4.75$ ( $P_f = \Phi(-\beta) \cong 1.0171 \cdot 10^{-6}$ )	MC	4.0839 ( $2.2143 \cdot 10^{-5}$ )	0.9309
	LHS-PAE	4.7258 ( $1.1458 \cdot 10^{-6}$ )	-
	Sobol	4.4943 ( $3.4896 \cdot 10^{-6}$ )	-
Sum2D $\beta = 3.35\sqrt{2} \cong 4.7376$ ( $P_f = \Phi(-\beta) \cong 1.0812 \cdot 10^{-6}$ )	MC	3.7503 ( $8.8321 \cdot 10^{-5}$ )	0.8103
	LHS-PAE	4.5663 ( $2.4821 \cdot 10^{-6}$ )	0.4678
	Sobol	4.7427 ( $1.0544 \cdot 10^{-6}$ )	-
Sin2D $\beta \cong 3.3425$ ( $P_f \cong 4.1508 \cdot 10^{-4}$ )	MC	2.9575 ( $1.5506 \cdot 10^{-3}$ )	0.5158
	LHS-PAE	3.4547 ( $2.7549 \cdot 10^{-4}$ )	0.3816
	Sobol	3.5977 ( $1.6049 \cdot 10^{-4}$ )	-

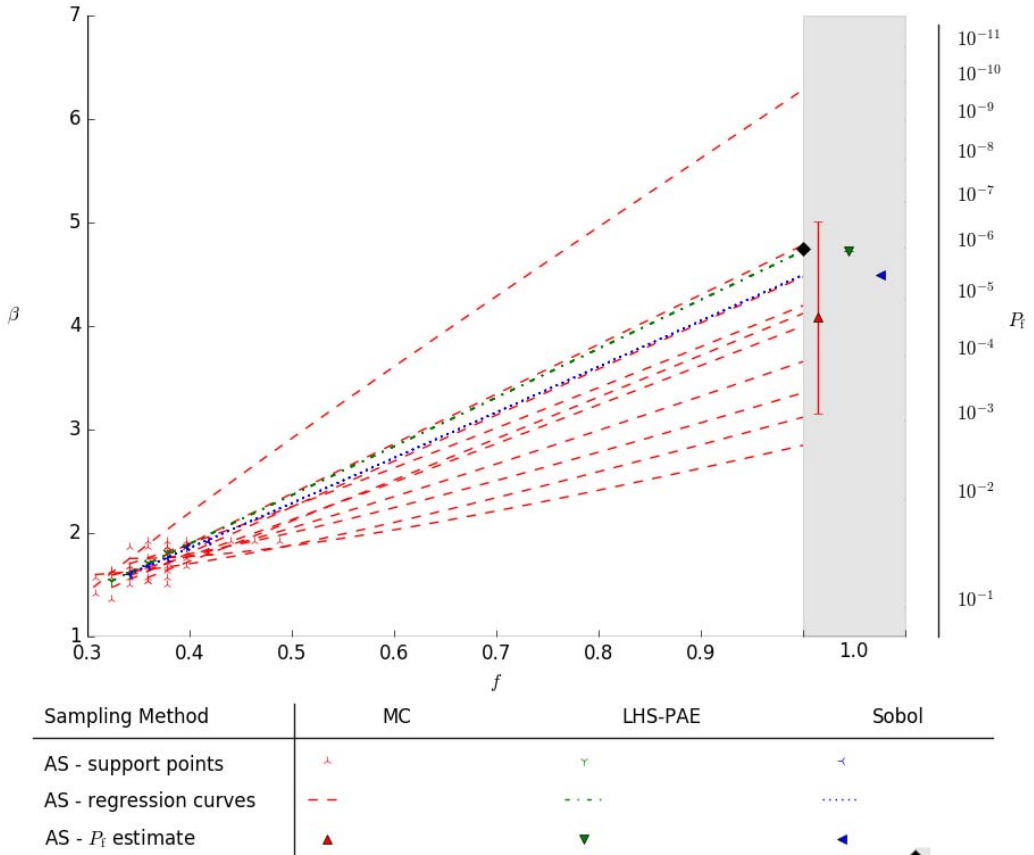


Fig.4: Sum1D function. The exact value (see Section 2) is marked ◆

## 6 CONCLUSION

The resulting graphs of all three functions (Fig. 4, 5, 6) show quite a large variance in the  $\beta$  estimates using AS combined with the MC sampling method. Although the MC method itself leads to unbiased estimates, combining it with AS may lead to bias, as can be seen in the graphs (the  $\beta$  estimates obtained by AS+MC for all three test functions are smaller than the exact  $\beta$  value). This is caused by the great variance in MC estimates in individual AS steps, which are already substantially modified (the  $\beta$  of the real problem is estimated from the modified problems in which the standard deviation of the input random variables is approximately three times as large as in the originally defined problem).

In the case of the combination of AS with LHS or QMC, the quality of the estimate is better not only in its mean but also its standard deviation. Based on the performed study, it cannot be clearly stated which of these two methods is more suitable. LHS designs have provided better results for the 1D-problem. In the case of the 2D test functions, one function was better estimated using LHS designs, whereas the estimation of the other was better with QMC designs. Sobol' sequences have zero standard deviation, which may also be a great advantage.

The present study shows that the selection of sampling scheme in AS technique may have a substantial impact on the average and variance of estimated failure probability. There is no doubt that the application of variance reduction techniques at the level of sampling from the underlying normalized normal distribution leads to decrease in variance of the extrapolated estimated reliability index  $\beta$ . The study encourages a more detailed research on the behaviour of presented combinations of methods for real civil engineering problems (both structural elements and systems). For example, not only the variance of the estimation is an issue. Also, the mean value of the extrapolation must converge

to the exact value of  $\beta$ . This is influenced mainly by the selection of the regression function which may be a difficult task for real civil engineering problems.

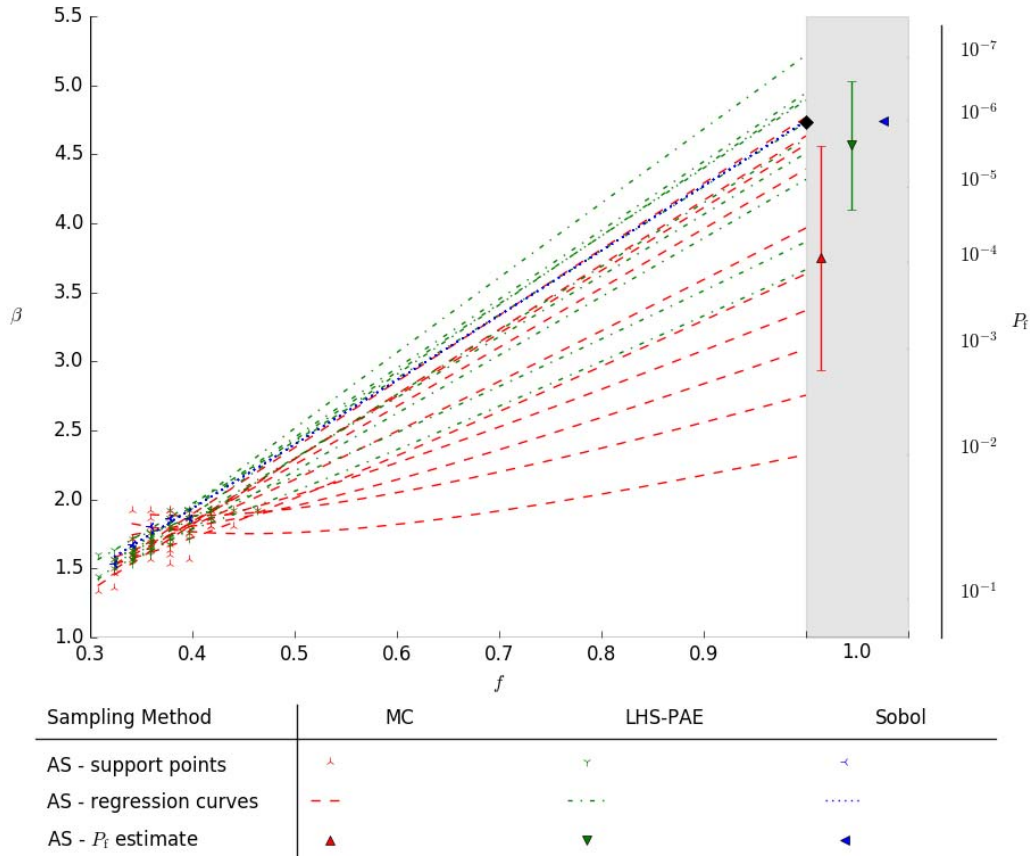


Fig.5: Sum2D function. The exact value (see Section 2) is marked  $\blacklozenge$

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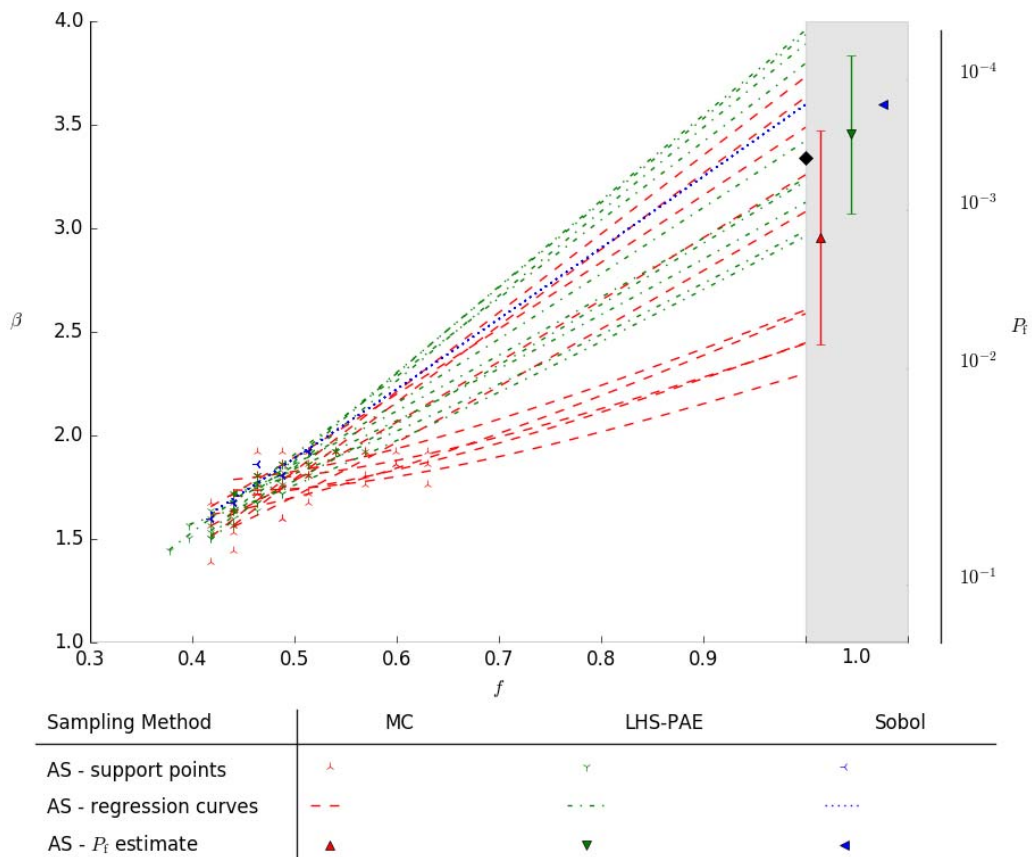


Fig.6: Sin2D function. The exact value (see Section 2) is marked ◆

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