

**Luboš ŠNIRC<sup>1</sup>, Alžbeta GRMANOVÁ<sup>2</sup>, Ján RAVINGER<sup>3</sup>****DYNAMIC POST-BUCKLING BEHAVIOR****Abstract**

Geometric non-linear theory has been used to describe the post-buckling behavior of slender web. Hamilton's principle in increments has been used. Examples of dynamic post-buckling behavior of slender web loaded in compression are presented. Influence of residual stresses to frequency of slender web is a base for non-destructive method of investigation of structures.

**Keywords**

Geometric non-linear problems, stability, dynamic post-buckling, vibration.

**1 INTRODUCTION**

Taking into account the stiffness and inertia forces, dynamic behavior of structures can be investigated. Dynamic investigation usually starts with an example of free vibration. It means to evaluate the natural frequency. The simplest stability problem of structures is buckling of a column. This problem can be arranged preparing the equilibrium conditions on a deformed structure. In general, however, for the evaluation of the stability problems strains should be evaluated for a deformed differential element what means to apply geometric non-linear theory.

Combination of dynamics and stability results in many problems: dynamic buckling, dynamic post buckling behavior, parametric resonance, etc. Introduction example – vibration of a column loaded in compression is simple but its investigation still represents a lot of problems.

The natural frequency can be measured by using rather simple equipment. The comparison of frequencies measured experimentally and evaluated numerically is the basis of non-destructive methods for investigation of structure properties. Generally, it can be said that in structural design stability effects have to be taken into consideration. These two ideas are the reason for our investigation of the combination of vibration and stability.

Leonard Euler [4] was probably the first scientist who had analyzed column stability problems. The former solutions are supposed to be the linear stability. It means that we suppose an ideal structure. The differences between theory and reality inspired researchers to search for more accurate models. Especially the slender web as the main part of thin-walled structure has significant post-buckling reserves and it is necessary to accept a geometric non-linear theory for their description. The problem of the vibration of the non-linear system was formulated by Bolotin [2]. Burgreen [3] analyzed the problem of the vibration of an imperfect column in early 50's of last

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century. Some valuable results have been achieved by Volmir [11]. Combination of dynamics and stability is still a subject of research all over the world [1, 5-6].

## 2 DYNAMIC POST-BUCKLING BEHAVIOR OF SLENDER WEB

### 2.1 Post-buckling behavior of slender web – displacement model

As it was already mentioned, slender web is the main constructional element of thin-walled structure [7-8]. If we assume an “ideal” slender web and a distribution of the in-plane stresses are not the function of the out-of plane (the plate) displacements, the problem leads to eigenvalues and eigenvectors. From the obtained eigenvalues elastic critical load can be evaluated and eigenvector characterizes the mode of buckling.

Post-buckling behavior can be assumed as follows (Fig. 1).

Displacements of the point of the middle surface are

$$\mathbf{q} = [u, v, w]^T \quad (1)$$

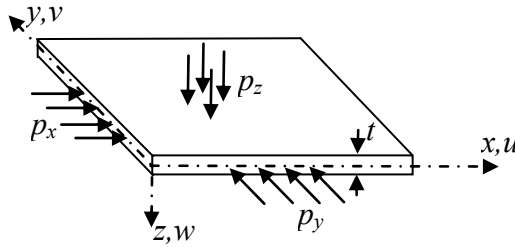


Fig.1: Notation of quantities of slender web

In the post-buckling behavior of the slender web the plate displacements are much larger than in-plane (web) displacements ( $w \gg u, v$ ) and so the strains are

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} w_{,x}^2 \\ w_{,y}^2 \\ 2w_{,x}w_{,y} \end{Bmatrix} - z \begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ w_{,xy} \end{Bmatrix} \quad (2)$$

where:

$z$  – is the coordinate of the thickness [m],

$x, y$  – partial derivations.

For the next investigation, slender web with initial deformations is assumed. Initial deformations are the plate types only.

$$\mathbf{q}_0 = [0, 0, w_0]^T \quad (3)$$

Due to that the initial strains are

$$\boldsymbol{\varepsilon}_0 = \frac{1}{2} \begin{Bmatrix} w_{0,x}^2 \\ w_{0,y}^2 \\ 2w_{0,x}w_{0,y} \end{Bmatrix} - z \begin{Bmatrix} w_{0,xx} \\ w_{0,yy} \\ 2w_{0,xy} \end{Bmatrix} \quad (4)$$



The “ $w$ ” represents the global displacements and “ $w_0$ ” is part related to the initial displacement. The linear elastic material has been assumed

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \boldsymbol{\sigma}_w \quad (5)$$

where:

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$E$  – is Young's modulus [Pa],

$\nu$  – is Poisson's ratio,

$\boldsymbol{\sigma}_w = [\sigma_{xw}, \sigma_{yw}, \tau_w]^T$  – are the residual stresses.

The global potential energy of the slender web is

$$U = U_i + U_e \quad (6)$$

where:

$U_i = \frac{1}{2} \int_V (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0)^T \boldsymbol{\sigma} dV$  – is potential energy of the internal forces,

$U_e = - \int_{\Gamma} (\mathbf{q} - \mathbf{q}_0)^T \mathbf{p} d\Gamma$  – is potential energy of the external forces,

where:

$V$  – is volume of the slender web,

$\Gamma$  – is in-plane surface.

The displacements are assumed as the product of the variational functions and the displacements parameters

$$\mathbf{q} = \mathbf{B} \cdot \mathbf{a} \quad (7)$$

The minimum of the global potential energy gives the system of conditional equation

$$\mathbf{K}_G(\mathbf{a}) \mathbf{a} = \mathbf{f} \quad (8)$$

where:

$\mathbf{K}_G$  – is stiffness matrix as the function of the displacement parameters – non-linear stiffness matrix,

$\mathbf{f}$  – is vector of the external load.



## 2.2 Post-buckling behavior of slender web loaded in compression - Example

For the simplification we suppose the square rectangular slender web loaded in compression simply supported all around.

We do not need to suppose the external load as the constant along the edge. But the external force must be defined as  $F = \int_0^b t \sigma dy$ . Consequently, the average stress can be defined as  $\sigma = \frac{F}{b.t}$ . For the approximate solution, we take displacement functions as

$$w = \alpha S_{x1} S_{y1}, w_0 = \alpha_0 S_{x1} S_{y1}, u = \beta_1 \left(1 - \frac{2x}{b}\right) + \beta_2 S_{x2} C_{y2} + \beta_3 S_{x2},$$

$$v = \gamma_1 \left(1 - \frac{2y}{b}\right) + \gamma_2 C_{x2} S_{y2} + \gamma_3 S_{y2},$$

where:

$$S_{xi} = \sin \frac{i\pi x}{b}, \dots C_{yi} = \cos \frac{i\pi y}{b}.$$

We have divided the variational parameters into:

- plate  $\alpha_D = \alpha$ ,
- in-plane  $\alpha_S = [\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3]^T$ . The in-plane displacements parameters are

$$\alpha_S = \frac{\pi}{16b} (\alpha^2 - \alpha_0^2) [\pi, 1, -1 + \nu, \pi, 1, -1 + \nu]^T$$

Introducing  $\sigma_E = \frac{\pi^2 E t^2}{12(1-\nu^2)b^2}$  (Euler's elastic critical stress), the dimensionless load as

$$\bar{\sigma} = \frac{\sigma'}{4\sigma_E} = \frac{\sigma'}{\sigma_{cr}}, \quad (\sigma_{cr} = 4\sigma_E) \text{ and the dimensionless parameters of the displacements function}$$

$$\bar{\alpha} = \frac{\alpha}{t}, \quad \bar{\alpha}_0 = \frac{\alpha_0}{t}, \text{ the result can be arranged into the final equation}$$

$$0,34125(\bar{\alpha}^2 - \bar{\alpha}_0^2) + 1 - \frac{\bar{\alpha}_0}{\bar{\alpha}} = \bar{\sigma} \quad (9)$$

The parameters  $\alpha$  and  $\alpha_0$  represent the amplitudes of the out of plate displacements of the slender web. Equation (9) is arranged in Fig. 2.

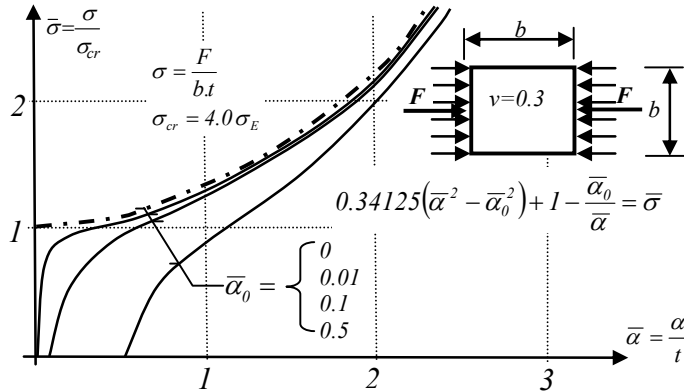


Fig.2: Post-buckling behavior of slender web loaded in compression



It is evident that slender web could be loaded above the level of the elastic critical load. Due to that “the post-buckling behavior” can be introduced. ***It has to be noted that the presented example represents an approximate solution.***

### 2.3 System of non-linear algebraic equations

First, we present a note related to the solution of geometric non-linear problems. We use (for example) the Ritz variational method. The functions of the displacements are sums of the products of the basic functions and the variational coefficients.

$$\mathbf{q} = \mathbf{B}.\alpha \quad (10)$$

These equations could be written in the mode

$$u, w \Rightarrow \alpha \uparrow 1 \quad (11)$$

The sign „ $\uparrow$ “ is used as the exponent. The elongations taking into account non-linear parts have the variational coefficients in quadrates and can be recorded as

$$\varepsilon = u_{,x} + \frac{1}{2} w_{,x}^2 - z.w_{xx} \Rightarrow \alpha \uparrow 2 \quad (12)$$

Assuming the linear elastic material, the stresses are in quadrates as well.

$$\sigma = E(\varepsilon - \varepsilon_0) \Rightarrow \alpha \uparrow 2 \quad (13)$$

The potential energy of the internal forces is a product of the elongations and the stresses, then, finally, the variational coefficients are of the fourth power

$$U_i = \frac{1}{2} \varepsilon^T . \sigma = (\alpha \uparrow 2).(\alpha \uparrow 2) = \alpha \uparrow 4 \quad (14)$$

The system of conditional equations may be arranged as a partial derivation according to the variational coefficients

$$\frac{\partial U}{\partial \alpha_i} = \dots = \alpha \uparrow 3 \quad (15)$$

Finally, we obtain the **system of cubic algebraic equations**.

A partial approval of our explanation can be seen in the example of the post bucking behavior of the slender web. (Part 2.2, Equation (9), where we have got the cubic algebraic equation.)

Note. In the example of the buckling of the column, the cubic terms have been eliminated. This “special case” is the consequence of the constant normal force along the column.

Let us continue with our former considerations. The system of linear algebraic equations can be arranged as a matrix (two dimensional area). The system of quadratic algebraic equations could be arranged as a three dimensional matrix. The cubic algebraic equations are a four dimensional matrix. We are not able to imagine the four dimensional matrix, but modern computers are able to compile it.

One typical property of the finite element method is a large number of parameters (many thousands). To arrange 1000 cubic algebraic equations represents in computer memory  $1000^4 = 1 \cdot 10^{12}$  real numbers and this is beyond possibilities. The way how to solve these non-linear systems has been found. The idea is to use the Newton-Raphson iteration without compilation of the system of non-linear (cubic) algebraic equations. It will be explained in the following parts.



## 2.4 Incremental formulation

As it has been already explained in the previous part, we are forced to arrange the iterative method. It can be prepared from the incremental formulation, and so we must prepare all the regulars in increments.

Note. All the rules for one dimensional problem (beams, columns) are prepared. For the solution of the two dimensional problems (webs, plates) the steps are similar. [6, 7]

As the first step, the increments and variations for the elongations must be prepared. If we have the linear function as

$$f\left(\frac{du}{dx}\right) = \frac{du}{dx} = u_{,x} \quad (16)$$

For the increments  $u + \Delta u$ , we get the increments of the function

$$\Delta f = f\left(\frac{d(u + \Delta u)}{dx}\right) - f\left(\frac{du}{dx}\right) = \frac{du}{dx} + \frac{d\Delta u}{dx} - \frac{du}{dx} = \frac{d\Delta u}{dx} = \Delta u_{,x} \quad (17)$$

We do the same steps for the non-linear function

$$f\left(\frac{dw}{dx}\right) = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \frac{1}{2}w_{,x}^2 \quad (18)$$

We have for the increment of this function

$$\begin{aligned} \Delta f &= f\left(\frac{d(w + \Delta w)}{dx}\right) - f\left(\frac{dw}{dx}\right) = \frac{1}{2}\left(\frac{d(w + \Delta w)}{dx}\right)^2 - \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \\ &= \frac{1}{2}\left(\left(\frac{dw}{dx}\right)^2 + 2\frac{dw}{dx} \cdot \frac{d\Delta w}{dx} + \left(\frac{d\Delta w}{dx}\right)^2\right) - \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \frac{dw}{dx} \cdot \frac{d\Delta w}{dx} + \frac{1}{2}\left(\frac{d\Delta w}{dx}\right)^2 = \\ &= w_{,x} \cdot \Delta w_{,x} + \frac{1}{2}\Delta w_{,x}^2 \end{aligned} \quad (19)$$

According to these rules the increment of the strain can be arranged as follows

$$\Delta \varepsilon_{,x} = \Delta u_{,x} + w_{,x} \cdot \Delta w_{,x} + \frac{1}{2}\Delta w_{,x}^2 - z \cdot \Delta w_{xx} \quad (20)$$

Then the variation of the increment of the elongation is prepare

$$\delta \Delta \varepsilon_{,x} = \delta \Delta u_{,x} + w_{,x} \cdot \delta \Delta w_{,x} + \delta \Delta w_{,x} \cdot \Delta w_{,x} - z \cdot \delta \Delta w_{xx} \quad (21)$$

## 2.5 Hamilton's principle

In this step, we prepare the rules for the dynamic process. In order to neglect the inertial forces, we get the static problems.

The Hamilton's principle means: in each time interval, the variation of the kinetic and potential energy and the variation of the work of the external forces is equal zero. This rule is valid for the increments as well:

$$\int_{t_0}^{t_1} \delta(\Delta T - \Delta U) dt + \int_{t_0}^{t_1} \delta \Delta W dt = 0 \quad (22)$$



where:

$$\Delta T = \int_V \frac{1}{2} \rho \Delta \mathbf{q}^T \Delta \mathbf{q} dV - \text{is increment of the kinetic energy,}$$

$$\Delta U = \int_V \left( \frac{1}{2} \Delta \varepsilon \Delta \sigma + \Delta \varepsilon \sigma \right) dV - \text{is increment of the potential energy of the internal forces,}$$

$$\Delta W = \int_V \Delta \mathbf{q}^T \cdot (\mathbf{p} + \Delta \mathbf{p}) dV - \text{is the increment of the work of the external forces,}$$

$t_0, t_I$  – are time intervals [s],

$\rho$  – mass density [kg/m<sup>3</sup>],

$V$  – is volume (in our case it is the volume of the beam – column) [m<sup>3</sup>],

$\mathbf{p}, \Delta \mathbf{p}$  – are external load and the increment of the external load.

The dots mean the time derivation. We assume the linear elastic material (Equation (5)). For the increments, we have  $\Delta \sigma = \mathbf{D} \Delta \varepsilon$ . In the case of the beam type of structures, the volume integration can be changed into the integration over the cross section and the integration over the length:  $A, I$  – the cross section area, the moment of inertia.

The longitudinal axis is situated into centre of the gravity of the cross section. We use the Ritz variational method

$$u = \mathbf{B}_S \cdot \alpha_S, \quad w = \mathbf{B}_D \cdot \alpha_D \quad (23)$$

We have the incremental model and the variational coefficients  $\alpha_S$  and  $\alpha_D$  are timeless functions. For the increments of the displacements functions, the independent basic variational functions can be used. The increments of the variational coefficients are the function of the time

$$\Delta u = \mathbf{B}_{S1} \cdot \Delta \alpha_S(t), \quad \Delta w = \mathbf{B}_{D1} \cdot \Delta \alpha_D(t) \quad (24)$$

Note. In some dynamic processes where there can be different boundary condition for the static behavior and for the vibration, it is useful to have different basic variational functions for the displacements and for the increment of the displacements.

Finally, equation (22) leads to the system of conditional equation. This system could be arranged into the mode

$$\begin{aligned} \mathbf{K}_{M-D} \Delta \alpha_D + \mathbf{K}_{INC-D} \Delta \alpha_D + \mathbf{K}_{INC-DS} \Delta \alpha_S + \mathbf{f}_{INT-D} - \mathbf{f}_{EXT-D} - \Delta \mathbf{f}_{EXT-D} &= \mathbf{0} \\ \mathbf{K}_{M-S} \Delta \alpha_S + \mathbf{K}_{INC-S} \Delta \alpha_S + \mathbf{K}_{INC-SD} \Delta \alpha_D + \mathbf{f}_{INT-S} - \mathbf{f}_{EXT-S} - \Delta \mathbf{f}_{EXT-S} &= \mathbf{0} \end{aligned} \quad (25)$$

where:

$$\mathbf{K}_{M-D} = \int_0^a \mathbf{B}_{D1}^T \rho A \mathbf{B}_{D1} dx - \text{is mass matrix of the “bending” displacements,}$$

$$\mathbf{K}_{INC-D} = \mathbf{K}_{INC-DL} + \mathbf{K}_{INC-DG} - \text{is incremental stiffness matrix of the bending,}$$

$$\mathbf{K}_{INC-DL} = \int_0^a \mathbf{B}_{D1XX}^T EI \mathbf{B}_{D1XX} dx - \text{is linear part,}$$

$$\mathbf{K}_{INC-DG} = \int_0^a \mathbf{B}_{D1X}^T EA \left( \frac{3}{2} w_{,x}^2 - \frac{1}{2} w_{0,x}^2 \right) \mathbf{B}_{D1X} dx - \text{non-linear part of the incremental stiffness matrix}$$

of the bending stiffness,



$\mathbf{K}_{INC-DS} = \int_0^a \mathbf{B}_{DIX}^T EA(w_{,x} + u_{,x} \cdot w_{,x}) \mathbf{B}_{SIX} dx$  – is incremental “bending – axial” stiffness matrix,

$\mathbf{f}_{INT-D} = \int_0^a \mathbf{B}_{DIXX}^T EI(w_{,xx} - w_{0,xx}) dx + \int_0^a \mathbf{B}_{DIX}^T EA(u_{,x} + \frac{1}{2} w_{,x}^2 u_{,x} - \frac{1}{2} w_{0,x}^2 u_{,x} + \frac{1}{2} w_{,x}^3 + w_{,x} u_{,x} - w_{,x} w_{0,x}^2) dx$  – the vector of the bending internal forces,

$\Delta \mathbf{f}_{EXT-D} = \int_0^a \mathbf{B}_{D1}^T \Delta \mathbf{p}_D dx$  – is increment of vector of the bending external forces,

$\mathbf{K}_{M-S} = \int_0^a \mathbf{B}_{S1}^T \rho A \mathbf{B}_{S1} dx$  – is mass matrix of the “axial” displacements,

$\mathbf{K}_{INC-S} = \int_0^a \mathbf{B}_{SX1}^T EA \mathbf{B}_{SX1} dx$  – is incremental stiffness matrix of the axial stiffness.

It can be proved that  $\mathbf{K}_{INC-SD} = \mathbf{K}_{INC-DS}^T$  – is incremental “axial – bending” stiffness matrix,

$\mathbf{f}_{INT-S} = \int_0^a \mathbf{B}_{SIX}^T EA \left( u_{,x} + \frac{1}{2} w_{,x}^2 - \frac{1}{2} w_{0,x}^2 \right) dx$  – is vector of the axial internal force,

$\mathbf{f}_{EXT-S} = \int_0^a \mathbf{B}_{S1}^T \mathbf{p}_S dx$  – is vector of the axial external forces,

$\Delta \mathbf{f}_{EXT-S} = \int_0^a \mathbf{B}_{S1}^T \Delta \mathbf{p}_S dx$  – is increment of the vector of the axial external forces.

It is evident that equation (25) represents the system of the differential equations of the second degree.

The axial and the bending displacement can be joined as

$$\Delta \alpha = \begin{Bmatrix} \Delta \alpha_D \\ \Delta \alpha_S \end{Bmatrix}, \quad \alpha = \begin{Bmatrix} \alpha_D \\ \alpha_S \end{Bmatrix}$$

The system of conditional equations (Equation (25)) could be written as

$$\mathbf{K}_M \Delta \alpha + \mathbf{K}_{INC} \Delta \alpha + \mathbf{f}_{INT} - \mathbf{f}_{EXT} - \Delta \mathbf{f}_{EXT} = \mathbf{0} \quad (26)$$

where:

$$\mathbf{K}_M = \left[ \begin{array}{c|c} \mathbf{K}_{M-D} & \\ \hline & \mathbf{K}_{M-S} \end{array} \right],$$

$$\mathbf{K}_{INC} = \left[ \begin{array}{c|c} \mathbf{K}_{INC-D} & \mathbf{K}_{INC-DS} \\ \hline \mathbf{K}_{INC-SD} & \mathbf{K}_{INC-S} \end{array} \right],$$

$$\Delta \mathbf{f}_{EXT} = \begin{Bmatrix} \Delta \mathbf{f}_{EXT-D} \\ \Delta \mathbf{f}_{EXT-S} \end{Bmatrix}, \quad \mathbf{f}_{EXT} = \begin{Bmatrix} \mathbf{f}_{EXT-D} \\ \mathbf{f}_{EXT-S} \end{Bmatrix}, \quad \mathbf{f}_{INT} = \begin{Bmatrix} \mathbf{f}_{IND-D} \\ \mathbf{f}_{INT-S} \end{Bmatrix}.$$



## 2.6 Hamilton's principle

The inertial forces can be neglected for the solution of the static behavior of the structure

$$\mathbf{K}_M \cdot \Delta \alpha \approx 0 \quad (27)$$

Note. In the case of the static behavior, except the Hamilton's principle, (Equation (22)) the principle of the minimum of the increment of the global potential energy can be applied.

The system of the differential equations (Equation (25)) will be changed into the system of the linear algebraic equation related to the increments of the displacements

$$\mathbf{K}_{INC} \Delta \alpha + \mathbf{f}_{INT} - \mathbf{f}_{EXT} - \Delta \mathbf{f}_{EXT} = 0 \quad (28)$$

If the problem is not established in the increments, but in the displacement parameters, we get the system of the cubic algebraic equations in the mode

$$\mathbf{f}_{INT} - \mathbf{f}_{EXT} = 0 \quad (29)$$

As previously explained in the introduction Part 2.3, this system of cubic algebraic equations cannot be compiled. (Note. This system can be arranged in some simple examples only.) Equation (28) is the basis for the incremental solution and for the Newton-Raphson iteration as well.

## 2.7 Incremental solution

We assume the system in equilibrium represented by the parameters of the displacements “ $\alpha$ ”. Then it is valid that

$$\mathbf{f}_{INT} - \mathbf{f}_{EXT} = 0 \quad (30)$$

The increment of the external load is obtained. The increments of the parameters of the displacements can be obtained from Equation (28)

$$\Delta \alpha = \mathbf{K}_{INC}^{-1} \Delta \mathbf{f}_{EXT} \quad (31)$$

The displacement parameters of the new level are

$$\alpha_D^i = \alpha_D + \Delta \alpha_D \quad (32)$$

## 2.8 Newton-Raphson iteration

We do not assume any system in equilibrium represented by the parameters of the displacements “ $\alpha^i$ ”. Then we have the vector of residuum

$$\mathbf{r}^i = \mathbf{f}_{INT} - \mathbf{f}_{EXT} \quad (33)$$

For the correction of the roots (displacement parameters), we assume the constant level of the external load ( $\Delta \mathbf{f}_{EXT} = 0$ ). Then it can be evaluated from equation (28)

$$\Delta \alpha^i = -\mathbf{K}_{INC}^{-1} \cdot \mathbf{r}^i \quad (34)$$



The new approximation of the displacement parameters is

$$\alpha^{i+1} = \alpha^i + \Delta\alpha^i \quad (35)$$

Equations (33, 35) represent the Newton-Raphson iteration. We have a large amount of parameters. For the completing the iterative process, it is necessary to use suitable norms. One of them could be

$$\|n\| = \frac{(\alpha^{i+1})^T \cdot \alpha^{i+1} - (\alpha^i)^T \cdot \alpha^i}{(\alpha^{i+1})^T \cdot \alpha^i} \leq 0.001; (0.0001) \quad (36)$$

Using the terminology of the Newton-Raphson iteration, we have

$$\mathbf{K}_{INC} = \mathbf{J} \quad (37)$$

The incremental stiffness matrix is the same as the Jacoby matrix of the Newton-Raphson iteration. The Jacoby matrix characterizes the tangent plane to the non-linear surface and is defined as

$$\mathbf{J}_{ij} \equiv \frac{\partial}{\partial \alpha_i} \mathbf{K}_{Gnel-ij}^* \quad (38)$$

where:

$\mathbf{K}_{Gnel}^*$  – is system of non-linear (in our case cubic) algebraic equations.

## 2.9 Bifurcation point

In the case of the non-linear problems, many results can be obtained represented by many paths (curves) illustrating relation of load versus the displacement parameters. Especially in the case of the stability problems, stable and unstable paths should be distinguished.

The global potential energy represents the surface. The local minimum of this surface is the point of stable path of the non-linear solution. From the theory of the quadratic surfaces for the local minimum, the Jacoby matrix (in our case, the incremental stiffness matrix) must be positively defined and all the principle minors must be positive as well

$$D = \left| \mathbf{K}_{INC} \right|_{\det} > 0, \quad D_k > 0 \quad (39)$$

If any condition of equation (39) is not satisfied, the path is unstable. The point between the stable and unstable paths is called **the bifurcation point**. In the bifurcation point, we have

$$D = \left| \mathbf{K}_{INC} \right|_{\det} = 0 \quad (40)$$

## 2.10 Vibration of the structure

The conditional equations have been arranged as a dynamic process. The static behaviour is taken as a partial problem. From the viewpoint of the dynamic, we consider only the problem of the vibration. We are able to evaluate the vibration of the structure in different load levels including the effects of initial imperfections.

We assume the structure in equilibrium and zero increment of the load

$$\Delta \mathbf{f}_{EXT} = \mathbf{0} \quad (41)$$



The system of conditional equations (Equation (25)) will be reduced

$$\mathbf{K}_M \Delta \alpha + \mathbf{K}_{INC} \Delta \alpha = \mathbf{0} \quad (42)$$

Related to the increments of the displacements parameters, this system represents a homogeneous differential equation with constant coefficient. The solution has the mode

$$\Delta \alpha = \Delta \bar{\alpha} \sin(\omega t) \quad (43)$$

where:

$\omega$  – is circular frequency.

Putting this into equation (42), we get

$$-\omega^2 \mathbf{K}_M \Delta \bar{\alpha} \sin(\omega t) + \mathbf{K}_{INC} \Delta \bar{\alpha} \sin(\omega t) = \mathbf{0} \quad (44)$$

The non-trivial solution leads to the problem of eigenvalues and eigenvectors

$$\left| \mathbf{K}_{INC} - \omega^2 \mathbf{K}_M \right|_{\det} = 0 \quad (45)$$

The eigenvalues represent the squares of circular frequencies, and eigenvectors are the parameters of the modes of the vibration.

Note. Incremental stiffness matrix includes level of the load, deformation of structure and initial imperfections as well.

### 3 STABILITY AND VIBRATION BEHAVIOUR OF SLENDER WEB

#### 3.1 Vibration of simple supported column loaded in compression

In Part 2.5, the derivation has been started by using the Hamilton's principle and generally prepared the conditional equation for the dynamic process. In Part 2.10, we have arranged the equations for the evaluation of the vibration.

Simple and interesting example is the vibration of the imperfect column. For the application of the action of the force, we must suppose one support as the hinge and the other support as the roller (the sliding support (Fig. 3)). (Note: The column is displayed in horizontal position.)

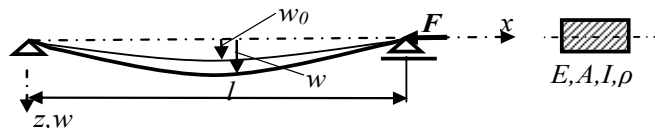


Fig.3: Simply supported column with initial displacement

The axial inertial forces are neglected and the displacement functions are  $w = \alpha_1 \sin \frac{\pi x}{l}$ ,  $w_0 = \alpha_0 \sin \frac{\pi x}{l}$ . The parameters of axial displacements are  $\alpha_2 = -\frac{F}{EA} - \frac{\pi^2}{l^2} (\alpha_1^2 - \alpha_0^2)$ ,  $\alpha_3 = -\frac{\pi}{8l} (\alpha_1^2 - \alpha_0^2)$ . The equation of the static behavior can be arranged in the form  $\bar{F} = \left(1 - \frac{\alpha_0}{\alpha_1}\right)$ , where  $\bar{F} = \frac{F}{F_{EU}}$ ,  $F_{EU} = \frac{\pi^2 EI}{l^2}$  is Euler's elastic critical force.



The incremental stiffness matrix is  $\mathbf{K}_{\text{INC}} = \frac{\pi^4 EI}{l^4} \frac{l}{2} - \frac{\pi^2}{l^2} \frac{l}{2} F$ . Putting this into equation (45)

obtained result is  $\omega^2 = \omega_0^2 (1 - \bar{F})$ . Square of circular frequency of the simply supported column is

$$\omega_0^2 = \frac{\pi^4 EI}{\rho A l^4} \quad (46)$$

We have obtained a trivial result of the linear relation of the square of the circular frequency and the internal force. It can be seen that during the free vibration the initial displacements do not affect the free vibration.

### 3.2 Vibration of simply supported column loaded fixed supports

Result represented by equation (46) in the case of the level of the load as the elastic critical load gives the zero frequency [10]. This is out of reality. For example, when the miner foreman knocks on the columns, low tone (low frequency) means the small force inside a column and column must be wedged. High tone (high frequency) means the high level of load and the additional columns must be used.

To improve the obtained result the following arrangement must be done (Fig. 4):

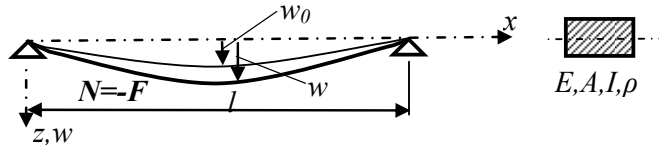


Fig.4: Simply supported column with initial displacement – fixed support

For the displacements and the initial displacements, we take  $w = \alpha_1 \sin(\pi x / l)$ ,  $w_0 = \alpha_0 \sin(\pi x / l)$ ,  $u = [x, \sin(2\pi x / l)] [\alpha_2, \alpha_3]^T$ . But for the increment of the displacement, we assume  $\Delta w = \Delta \alpha_1 \sin(\pi x / l)$ ,  $\Delta u = \Delta \alpha_3 \sin(2\pi x / l)$ . Now, different basic variational functions are used for the displacements and for the initial displacements, finally the incremental stiffness matrix is  $\mathbf{K}_{\text{INC}} = \frac{\pi^4 EI}{l^4} \frac{l}{2} - \frac{\pi^2}{l^2} \frac{l}{2} F + EA \frac{\pi^4}{l^4} \frac{l}{2} \frac{\alpha_1^2}{2}$ . Then we get the expression for the square of the circular frequency

$$\omega^2 = \omega_0^2 \left( 1 - \bar{F} + \frac{1}{2} \frac{\alpha_1^2}{r^2} \right) \quad (47)$$

where:

$r = \sqrt{\frac{I}{A}}$  is the radius of inertia.

Thus, the result close to reality has been obtained. (Fig. 5) The displacement parameter „ $\alpha_1$ “ is the function of the initial displacement and the level of the load. It means that the initial displacement enters the problem. If the load limits the level of the elastic critical load, the displacement and the frequency limit the infinity.



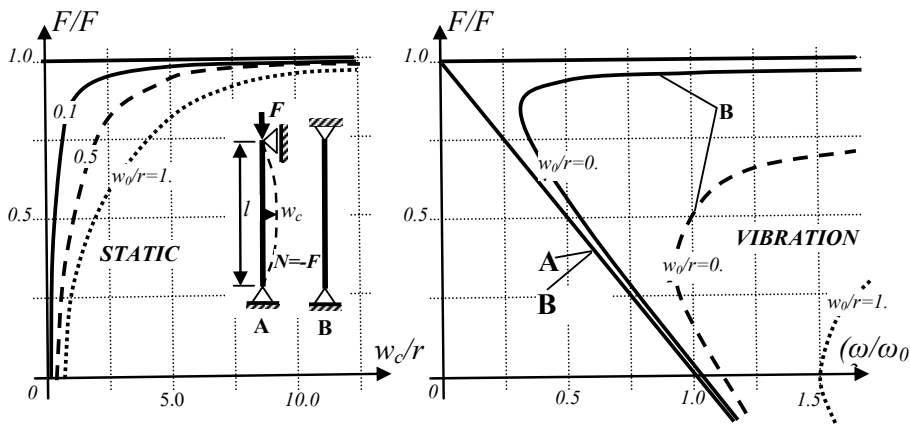


Fig.5: Stability and vibration of imperfect column

This example represents an advantage of the separation of the basic variational functions for the displacements and for the increments of the displacements.

### 3.3 Initial displacement as the second mode of buckling

Partial interesting problem is the influence of the mode of the initial displacement. In the previous part, we have supposed the initial displacement in the same mode as the first buckling mode (the mode of buckling related to the lowest elastic critical load). Due to that to obtain the solution by the analytical way was rather easy.

Figure 6 presents solution of buckling and vibration of column when the initial displacement has mode related to the second mode of buckling.

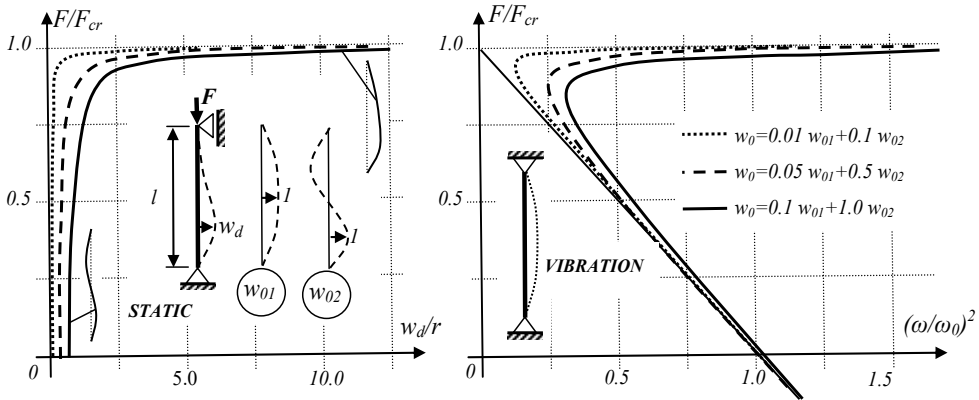


Fig.6: Stability and vibration of imperfect column with the initial displacement as the second mode of buckling

Note. A lot of examples have been solved using the FEM. The obtained results can be presented in the dimensionless mode.

These results enable us to note some peculiarities. Even the initial displacement has the same mode as the second mode of the buckling (*“the mode 2”*), the collapse mode of the column is *“the mode 1”*. The lowest elastic critical load is the maximum load. The mode of the vibration is *“the mode 1”* in all cases.



### 3.4 Experimental verification

Presented theoretical solutions are pointing to a substantial difference in the vibration of the beam at the moment when the critical load is reached. Considering sliding supports, the frequency should be zero. When supports are fixed, the frequency limits in infinity. This curiosity has been verified by an experiment.

The equipment for experimental verification of stability and vibration of beams loaded by pressure is shown in figure 7 and figure 8.

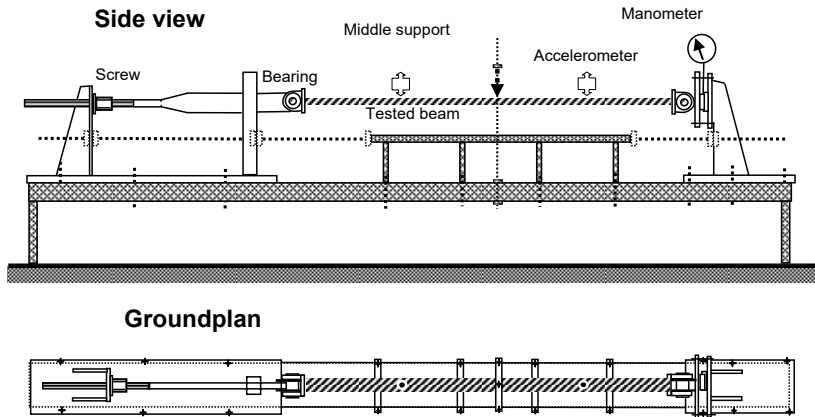


Fig.7: Scheme of the test set-up

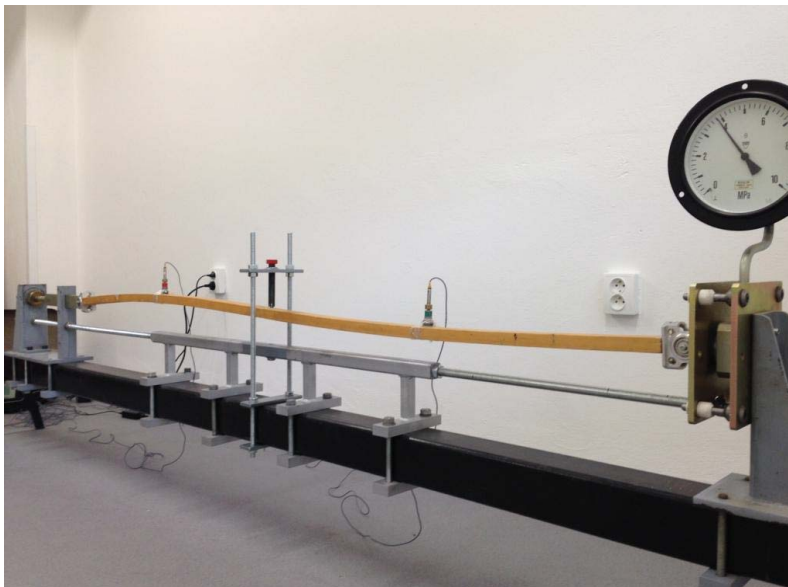


Fig.8: General view of the test

The force (the load) is produced through the screw with a slight gradient (gradient 1.5 mm, average 30 mm), it means the load with the controlled deformation. The hinges are created by ball



bearings in the jaw. The force is measured by manometer. The deflections are measured by mechanical displacement transducers fixed to the supporting steel structure. During measuring the frequency, the mechanical transducers are taken out and the accelerometer is attached.

Before the presentation of results, it is appropriate to make a note for specification of the mass matrixes due to end bearing as shown in figure 9.

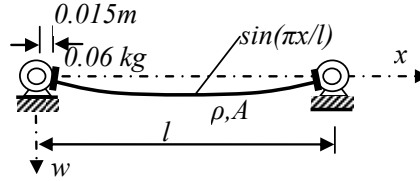


Fig.9: Effects of end-bearing of beam to the mass matrix

The mass matrix taking into account the effect of the end bearing will be  $\mathbf{K}_M = \rho A \frac{l}{2} + 2 * 0.06 * \sin \frac{\pi * 0.015}{l}$  where the length of the beam is given in meters.

This effect of the end-bearing is dependent on the mass of the beam and is small (less than 1.5 %). To verify the dependence between the pressure force and frequency, the beams made of various types of materials have been analyzed.

#### Steel hollow section profile CHS 30/15/1.5 mm

In the case of steel, the value of modulus of elasticity and the mass density are constant. When the exact dimensions of closed sections were measured, small problem occurred in measuring wall thickness.

The dimensions have been specified by measuring the weight of the profile. The rounded corners were considered in specification of cross-sectional characteristics. For further evaluation the following values were used.

$$CHS\ 29.9/14.8/1.53; A = 121.4\ mm^2; I = 4286.0\ mm^4; i = 5.94\ mm; l = 1450\ mm$$

$$E = 210000\ MPa; \mu = 7850\ kg / m^3; F_{cr} = 4225.1\ N; \omega_0 = 144.2\ s^{-1}$$

#### Timber beams

The modulus of elasticity of wood is an open question in the analyses of timber beams. In the presented measurements the critical load is identified at the moment of the increasing of the deformation without the increase of the force. Since the cross-sectional characteristics (the cross section, the moment of inertia) as well as the length of the beam have been known, using the Euler's elastic critical force, the modulus of elasticity can be evaluated. By measuring the weight of the profile, the mass density of wood has been easily and accurately evaluated. Subsequently, the natural circular frequency has been evaluated and two timber beams investigated.

$$Timber\ beam\ 47/47\ mm; A = 2209\ mm^2; I = 406640\ mm^4; i = 13.57\ mm; l = 2040\ mm$$

$$E = 10200\ MPa; \rho = 472\ kg / m^3; F_{cr} = 9836.7\ N; \omega_0 = 147.3\ s^{-1}$$

$$Timber\ beam\ 42/32\ mm; A = 1344\ mm^2; I = 114888\ mm^4; i = 9.24\ mm; l = 1650\ mm$$

$$E = 9750\ MPa; \rho = 454\ kg / m^3; F_{cr} = 4060.8\ N; \omega_0 = 1154.9\ s^{-1}$$



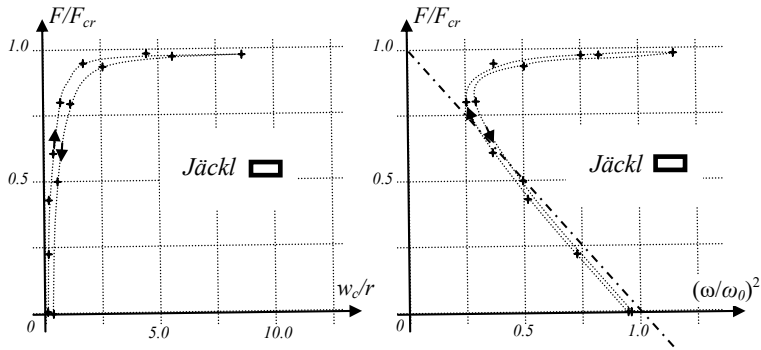


Fig.10: Results from measurements of the steel hollow thin-walled section – CHS

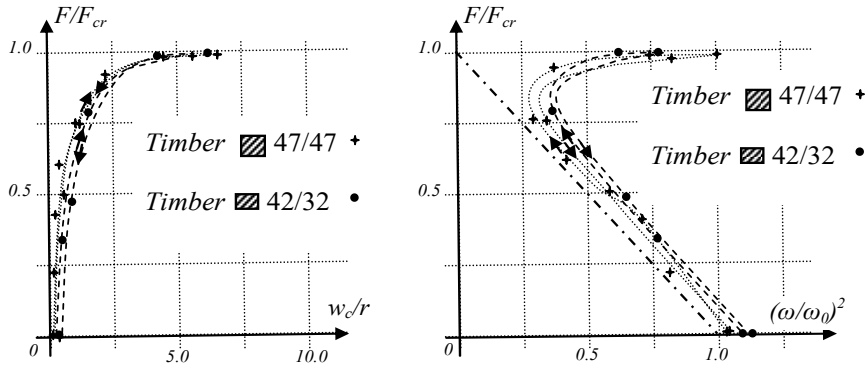


Fig.11: Results from the measurements of the timber beams

The presented results confirmed undoubtedly a phenomenon that the frequency of the beam increases when the pressure force is near the critical level.

#### Continuous beam

Figures 12 and 13 present the dimensions of investigated continuous beam. Loading was implemented by steps.

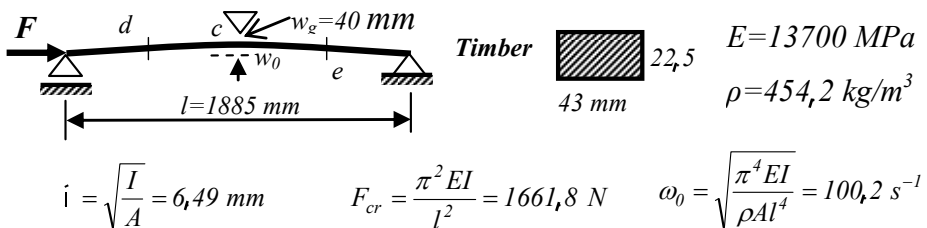


Fig.12: Dimension of continuous timber beam



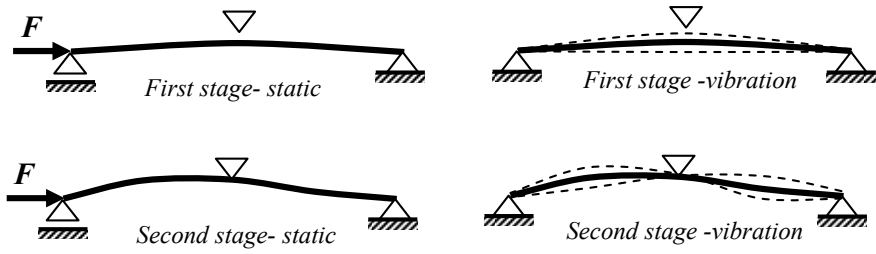


Fig.13: Two states of continuous beam behavior

Figures 14 and 15 present obtained results arranged in dimensionless form. Computes program had to be special improved for numerical evaluation of this example.

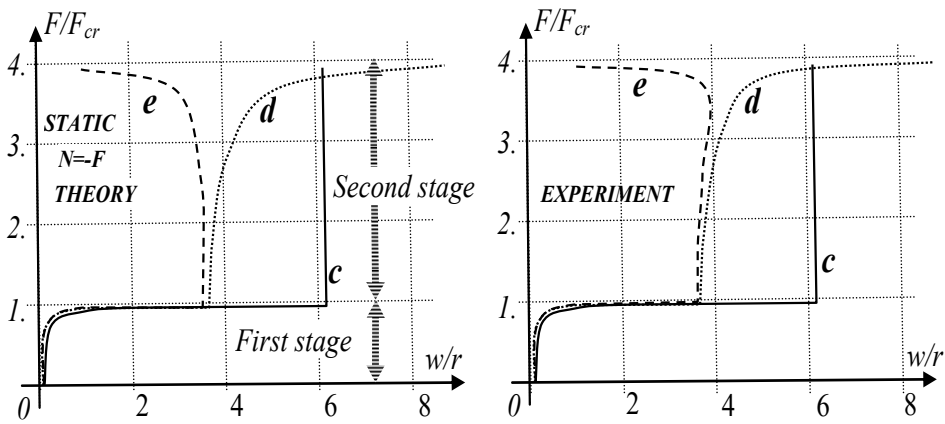


Fig.14: Static behaviour of continues beam

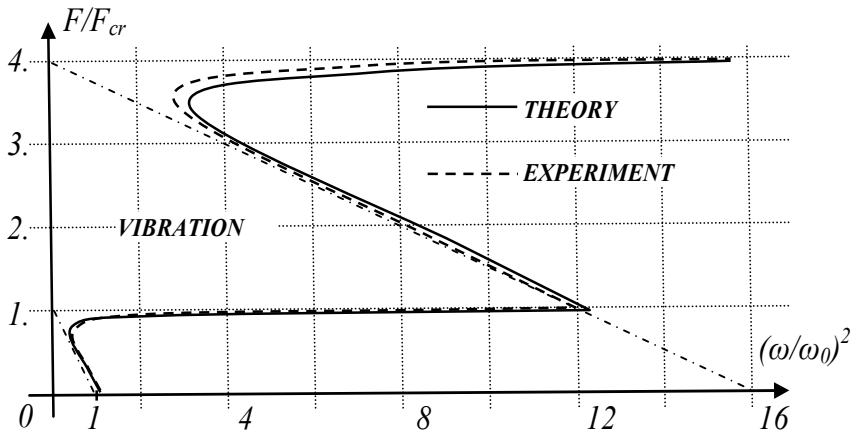


Fig.15: Load versus square of circular frequency



## 4 VIBRATION AND RESIDUAL STRESSES

### 4.1 Vibration of simple supported column loaded in compression

In Part 2.5, the derivation residual stresses ( $\sigma_w$  – equation (5)) are typical in the welded steel structures. Taking these stresses into increment of global potential energy and after doing variation we get term as product of increment of variation of derivation of displacement functions and residual stresses

$$\dots \int_V \delta \Delta \varepsilon \cdot \sigma_w dV = \dots \int_V \left( \delta \Delta u_{,x} + u_{,x} \cdot \delta \Delta u_{,x} + w_{,x} \cdot \delta \Delta w_{,x} + \delta \Delta w_{,x} \cdot \Delta w_{,x} - z \cdot \delta \Delta w_{xx} \right) \sigma_w dV \quad (48)$$

In the case of the beam type of structures, the volume integration can be changed into the integration over the cross section and the integration over the length.

$$\dots \int_0^a \left( \delta \Delta u_{,x} + u_{,x} \cdot \delta \Delta u_{,x} + w_{,x} \cdot \delta \Delta w_{,x} + \delta \Delta w_{,x} \cdot \Delta w_{,x} \right) \int_A \sigma_w dA - \delta \Delta w_{xx} \int_A z \cdot \sigma_w dA \Bigg) dx =$$

The residual stresses must be in equilibrium in the given cross section

$$\int_A \sigma_w dA = 0, \quad \int_A z \cdot \sigma_w dA = 0 \quad \Rightarrow \quad \dots \int_V \delta \Delta \varepsilon \cdot \sigma_w dV = 0 \quad (49)$$

It is evident that *the residual stresses in the case of the beam structures have no influence on the circular frequency.*

Note. In the case of the statically indeterminate structure, equation (48) is not valid and the residual stresses could have the influence on the vibration.

There is much different situation in the case of the plate structures. In this case, the volume integration is divided into the integration over the thickness and the integration over the neutral surface. The integration of the residual stresses over the thickness is not zero and thus,

$$\dots \int_V \delta \Delta \varepsilon \cdot \sigma_w dV = \int_{\Gamma} \left( \int_{-t/2}^{t/2} \delta \Delta \varepsilon \cdot \sigma_w dz \right) d\Gamma = \int_{\Gamma} \left( \delta \Delta \varepsilon \int_{-t/2}^{t/2} \sigma_w dz \right) d\Gamma \neq 0$$

Finally, *in the case of the plate structures, the residual stresses have an influence on the circular frequency.*

Effect of residual stresses on circular frequency has been proved by experiment [9]. (Fig. 16). Some results are presented in figure 17 and figure 18.



Fig.16: General view of experimental arrangement for the test of thin-walled panel



## 6 CONCLUSIONS

Presented theory and results prove the influence of the natural frequency on the level of the load, on the geometrical imperfections and the residual stresses, too. This knowledge can be used as an inverse idea. Measuring of the natural frequencies provides a picture of the stresses and imperfections in a thin-walled structure. One idea how we can investigate the structure is presented in figure 19. Many times we are not able to measure the whole structure (global vibration) but even measuring local parts of structure (local vibration) can give us valuable results.

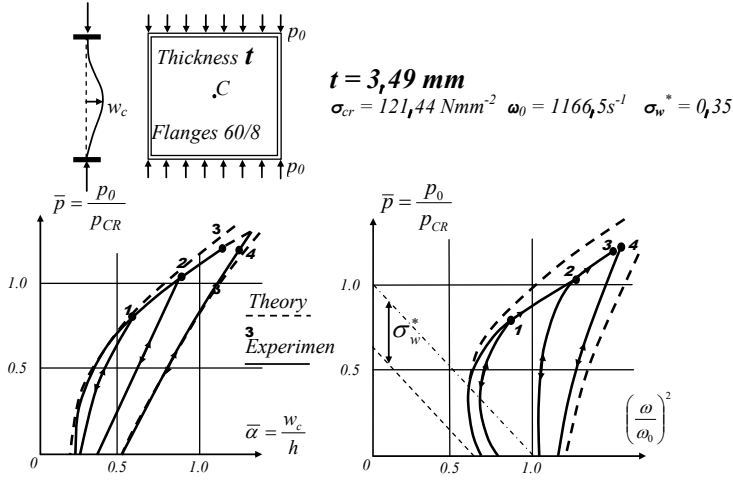


Fig.17: Comparison of theoretical and experimental results for the steel panel with  $t=3.49 \text{ mm}$  thick web

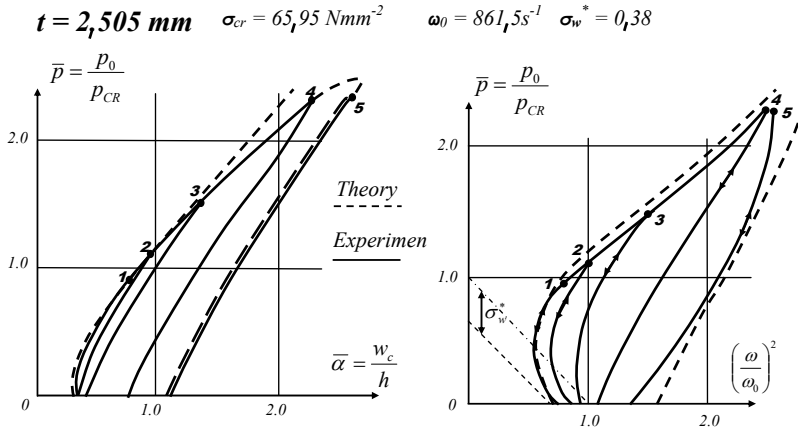


Fig.18: Comparison of theoretical and experimental results for the steel panel with  $t=2.505 \text{ mm}$  thick web

It is true that the relation of frequencies versus stresses and imperfections represents a sophisticated theory, but it is unlikely an obstacle for further investigation.



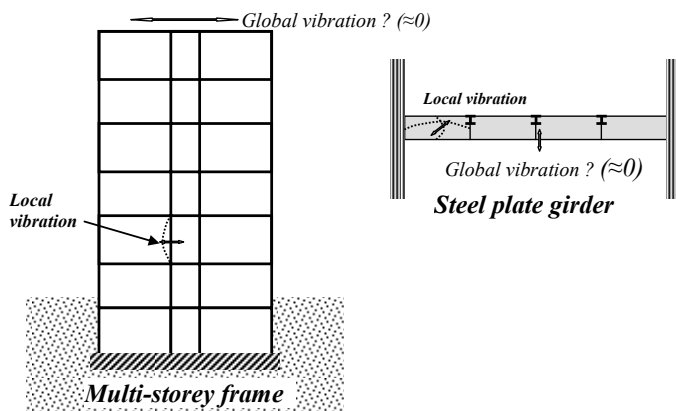


Fig.19: Scheme for non-destructive investigation of structure properties

## ACKNOWLEDGMENT

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