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VIBRATIONS AND DEFORMATIONS OF MODERATELY THICK PLATES IN STOCHASTIC FINITE ELEMENT METHOD

Abstract

The paper deals with some chosen aspects of stochastic dynamical analysis of moderately thick plates. The discretization of the governing equations is described by the finite element method. The main aim of the study is to provide the generalized stochastic perturbation technique based on classical Taylor expansion with a single random variable.

Keywords

Stochastic perturbation technique, finite element method, plates, dynamics.

1 INTRODUCTION

In the paper the finite element method has been applied to quantify the effects of random inputs on plates deflection with taking into account sheering forces. Dynamic equation of motion and free vibration equation has been introduced [6, 7]. The so-called stochastic finite element method has been used on the basic of the 2nd-order perturbation method [1-5]. This non-statistical approach is numerically much more efficient than a statistical approach, such as Monte Carlo simulation. A major advantage of the statistical finite element approach is that only the first two moments need to be known. Moreover a large number of samples are required in statistical approaches.

2 FORMULATION OF THE PROBLEM

2.1 Moderately thick plates - governing equations

This chapter will first present basic equations governing the problem. Further, the equations of a finite element method will be discussed. Let first introduce differential operators

$$\boldsymbol{d}_1^T = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \tag{1}$$

$$\boldsymbol{d}_2^T = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix} \tag{2}$$

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Dynamic equilibrium equations can be describe as follows

$$\boldsymbol{d}_2^T\boldsymbol{Q} + p_z = -\left(\rho h\frac{\partial^2 w}{\partial t^2} + \mu_1 h\frac{\partial w}{\partial t}\right) \tag{3}$$

$$\boldsymbol{d}_1^T\boldsymbol{B}_m - \boldsymbol{Q} + \boldsymbol{b}_m = -\left(\rho\frac{h^3}{12}\frac{\partial^2 \boldsymbol{\upsilon}}{\partial t^2} + \boldsymbol{\mu}\frac{h^3}{12}\frac{\partial \boldsymbol{\upsilon}}{\partial t}\right) \tag{4}$$

where $\boldsymbol{Q} = \{Q_x, Q_y\}$ is a shearing forces vector, $\boldsymbol{B}_m = \{B_{mx}, B_{my}, B_{mxy}\}$ is a bending and twisting moments vector, $\boldsymbol{b}_m = \{b_{mx}, b_{my}\}$ is a external load vector (bending moments), p_z is a external pressure, ρ is a density, μ_1 is a damping parameter $\boldsymbol{\mu}$ is a damping matrix, h is a height of plate, t - time, $\boldsymbol{\upsilon} = \{\upsilon_x, \upsilon_y\}$ is an angle of rotation vector and w is a vertical displacement. Geometric equations have the form

$$\boldsymbol{\beta} = \boldsymbol{d}_2 w + \boldsymbol{\upsilon}\ , \qquad \boldsymbol{\kappa} = \boldsymbol{d}_1\boldsymbol{\upsilon} \tag{5}$$

where $\boldsymbol{\kappa}$ is a curvatures vector, $\boldsymbol{\beta}$ is an angle of shear strain.

Assuming that

$$\boldsymbol{D} = D\boldsymbol{n}\ , \qquad \boldsymbol{n} = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{6}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}\ , \qquad H = \frac{5}{6}Gh \tag{7}$$

where ν is a Poisson's ratio, E is a modulus of elasticity in tension and compression, G is a modulus of elasticity in shear, finally may be written constitutive equations

$$\boldsymbol{B}_m = \boldsymbol{D\kappa} \tag{8}$$

$$\boldsymbol{Q} = H\boldsymbol{\beta} + \frac{1}{6}\boldsymbol{b} \tag{9}$$

Above mentioned set of thirteen equations (3-5, 8-9) may be combined into concise form of three differential equations with unknown displacements

$$\boldsymbol{d}_1^T\left(\boldsymbol{D}\boldsymbol{d}_1\boldsymbol{\upsilon}\right) - H\boldsymbol{\upsilon} - H\boldsymbol{d}_2 w + \frac{5}{6}\boldsymbol{b} = -\left(\rho\frac{h^3}{12}\frac{\partial^2 \boldsymbol{\upsilon}}{\partial t^2} + \boldsymbol{\mu}\frac{h^3}{12}\frac{\partial \boldsymbol{\upsilon}}{\partial t}\right) \tag{10}$$

$$H\boldsymbol{d}_2^T\boldsymbol{\upsilon} + H\boldsymbol{d}_2^T\boldsymbol{d}_2 w + p_z + \frac{1}{6}\boldsymbol{d}_2^T\boldsymbol{b} = -\left(\rho h\frac{\partial^2 w}{\partial t^2} + \mu_1 h\frac{\partial w}{\partial t}\right) \tag{11}$$

Expression of an approximation of the displacement field is described in the following way

$$\boldsymbol{f} = \{w, \boldsymbol{\upsilon}\} = \boldsymbol{N}(\xi,\eta)\boldsymbol{q}^{(e)} \tag{12}$$

where $\boldsymbol{q}_i^{(e)} = \{w_i, \upsilon_{xi}, \upsilon_{yi}\}$ and $\boldsymbol{N}(\xi,\eta)$ is a shape functions (in element local coordinates ξ,η) matrix. External loads can be combined into one vector $\boldsymbol{p} = \{p_z, \boldsymbol{b}\}$. The same way is used to combine internal forces $\boldsymbol{\sigma} = \{\boldsymbol{B}_m, \boldsymbol{Q}\} = \boldsymbol{E}_{gl}\boldsymbol{\varepsilon}$ where

$$\boldsymbol{E}_{gl} = \begin{bmatrix} \boldsymbol{D} & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H \end{bmatrix} \tag{13}$$

Deformation vector is described as follows

$$\boldsymbol{\varepsilon} = \boldsymbol{dNq}^{(e)} = \boldsymbol{Bq}^{(e)} \tag{14}$$

where differential operators (1, 2) was combined to one matrix

$$\boldsymbol{d}^T = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{a}\frac{\partial}{\partial\xi} & \frac{1}{b}\frac{\partial}{\partial\eta} \\ \frac{1}{a}\frac{\partial}{\partial\xi} & 0 & \frac{1}{b}\frac{\partial}{\partial\eta} & 1 & 0 \\ 0 & \frac{1}{b}\frac{\partial}{\partial\eta} & \frac{1}{a}\frac{\partial}{\partial\xi} & 0 & 1 \end{bmatrix} \tag{15}$$

Finally, after same manipulations, and with Galerkin method or variational approach, finite element method equation is formulated

$$\iint_S \boldsymbol{N}^T \boldsymbol{\rho} \boldsymbol{N} dS \ddot{\boldsymbol{q}} + \iint_S \boldsymbol{N}^T \boldsymbol{\mu} \boldsymbol{N} dS \dot{\boldsymbol{q}} + \iint_S \boldsymbol{B}^T \boldsymbol{E}_{gl} \boldsymbol{B} dS \boldsymbol{q} = \iint_S \boldsymbol{N}^T \boldsymbol{p} dS \tag{16}$$

where

$$\boldsymbol{\rho} = \begin{bmatrix} \rho h & 0 & 0 \\ 0 & \rho\frac{h^3}{12} & 0 \\ 0 & 0 & \rho\frac{h^3}{12} \end{bmatrix} \tag{17}$$

Equation (16) can be written in compact form

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{Kq} = \boldsymbol{F} \tag{18}$$

Second term of the left-hand side of equation (18) with damping matrix $\boldsymbol{\mu}$ can be described as Rayleigh damping with pre-defined coefficients α_1 and α_2

$$\boldsymbol{C} = \alpha_1 \boldsymbol{M} + \alpha_2 \boldsymbol{K} \tag{19}$$

2.1 Perturbation method - equations of motion and free vibrations

The basic concept of second moment perturbation method (SMPM) is descended from the linear transform of a random variable described in term of a power series expansion [1-3]. Let us consider a vector $\boldsymbol{a} = \{a_r\}$, $r = 1, 2, \ldots, \hat{r}$, are assumed to be time-independent random variables, specified by the first two associated central moments – means $\bar{\boldsymbol{a}} = \{\bar{a}_r\}$ and cross-covariances $Cov(a_r, a_s)$; $r, s = 1, 2, \ldots, \hat{r}$. In comparison with conventional statistical approaches, Monte Carlo simulation for instance, the drawbacks of the non-statistical SMPM are that (i) random variables $\{x_i\}$ must satisfy the conditions for small fluctuation and for continuity at $\{\bar{a}_r\}$, and (ii) only first two probabilistic moments can be given on output. On the other hand, advantages of SMPM are significant, since (a) the assumption of the normal distribution (even homogeneity) for $\{x_i\}$ is not necessarily needed, (b) only the first two moment for $\{a_r\}$ are required on input, and (c) with the same-order accuracy only $o(\hat{r})$ equation system to be solved in SMPM when compared with $o(\hat{r}^3)$ corresponding systems sampled in Monte Carlo simulation.

Hierarchical system for the multidegree-of-fredom system describing structural dynamic response with system mass M, damping C, stiffness matrix K, displacement $q = \{\boldsymbol{v}_i, w_i\}$ and load vector $\boldsymbol{F}$ is

$$\boldsymbol{M}^0\ddot{\boldsymbol{q}}^0 + \boldsymbol{C}^0\dot{\boldsymbol{q}}^0 + \boldsymbol{K}^0\boldsymbol{q}^0 = \boldsymbol{F}^0 \tag{20}$$

$$\boldsymbol{M}^0\ddot{\boldsymbol{q}}^{,r} + \boldsymbol{C}^0\dot{\boldsymbol{q}}^{,r} + \boldsymbol{K}^0\boldsymbol{q}^{,r} = \boldsymbol{F}^{,r} - \left[\boldsymbol{M}^{,r}\ddot{\boldsymbol{q}}^0 + \boldsymbol{C}^{,r}\dot{\boldsymbol{q}}^0 + \boldsymbol{K}^{,r}\boldsymbol{q}^0\right] \qquad r = 1,\ 2,\ldots,\hat{r} \tag{21}$$

$$\boldsymbol{M}^0\ddot{\boldsymbol{q}}^{(2)} + \boldsymbol{C}^0\dot{\boldsymbol{q}}^{(2)} + \boldsymbol{K}^0\boldsymbol{q}^{(2)} = \sum_{r,s=1}^{\hat{r}}\Big(\boldsymbol{F}^{,rs} +$$

$$-2\left(\boldsymbol{M}^{,r}\ddot{\boldsymbol{q}}^{,s} + \boldsymbol{C}^{,r}\dot{\boldsymbol{q}}^{,s} + \boldsymbol{K}^{,r}\boldsymbol{q}^{,s}\right) - \left(\boldsymbol{M}^{,rs}\ddot{\boldsymbol{q}}^0 + \boldsymbol{C}^{,rs}\dot{\boldsymbol{q}}^0 + \boldsymbol{K}^{,rs}\boldsymbol{q}^0\right)\Big)Cov\left(a_r, a_s\right) \tag{22}$$

where $\boldsymbol{q}^{(2)} = \sum_{r,s=1}^{\hat{r}}\left(\boldsymbol{q}^{,rs}Cov\left(a_r, a_s\right)\right)$, the symbols $(\circ)^0$, $(\circ)^{,r}$ and $(\circ)^{,rs}$ denote the values of the zero, first and mixed second partial derivatives $(\circ)$ with respect to $\{a_r\}$ at $\{\bar{a}_r\}$, respectively.

Hierarchical system for the multidegree-of-fredom system describing the generalized eigenproblem of free vibrations with system mass M, stiffness K, eigenvalue $\boldsymbol{\Omega}$ and eigenvector $\boldsymbol{\phi}$ matrices have the form

$$\left[\boldsymbol{K}^0 - \boldsymbol{\Omega}^0\boldsymbol{M}^0\right]\boldsymbol{\phi}^0 = \boldsymbol{0} \tag{23}$$

$$\left[\boldsymbol{K}^0 - \boldsymbol{\Omega}^0\boldsymbol{M}^0\right]\boldsymbol{\phi}^{,r} = -\left[\boldsymbol{K}^{,r} - \boldsymbol{\Omega}^{,r}\boldsymbol{M}^0 - \boldsymbol{\Omega}^0\boldsymbol{M}^{,r}\right]\boldsymbol{\phi}^0 \qquad r = 1,\ 2,\ldots,\hat{r} \tag{24}$$

$$\left[\boldsymbol{K}^0 - \boldsymbol{\Omega}^0\boldsymbol{M}^0\right]\boldsymbol{\phi}^{(2)} = \sum_{r,s=1}^{\hat{r}}\Big(\left[\boldsymbol{K}^{,rs} - \boldsymbol{\Omega}^{,rs}\boldsymbol{M}^0 - 2\boldsymbol{\Omega}^{,r}\boldsymbol{M}^{,r} - \boldsymbol{\Omega}^0\boldsymbol{M}^{,rs}\right]\boldsymbol{\phi}^0 +$$

$$-2\left[\boldsymbol{K}^{,r} - \boldsymbol{\Omega}^{,r}\boldsymbol{M}^0 - \boldsymbol{\Omega}^0\boldsymbol{M}^{,r}\right]\boldsymbol{\phi}^{,r}\Big)Cov\left(a_r, a_s\right) \tag{25}$$

where $\boldsymbol{\phi}^{(2)} = \sum_{r,s=1}^{\hat{r}}\left(\boldsymbol{\phi}^{,rs}Cov\left(a_r, a_s\right)\right)$.

3 EXAMPLE

In this example the deflection of simply supported square plate and clamped square plate with random thickness subjected to a concentrated center load are examined.

Tab. 1: Ten first frequencies for plate

Mode number	**Simply supported**			**Clamped**		
	Circular frequency [rad/sec]	**Frequency [cycl/sec]**	**Period [sec]**	**Circular frequency [rad/sec]**	**Frequency [cycl/sec]**	**Period [sec]**
1	108.6	17.29	5.78e-02	199.4	31.73	3.15e-02
2	272.6	43.38	2.30e-02	408.9	65.07	1.53e-02
3	272.6	43.38	2.30e-02	408.9	65.07	1.53e-02
4	438.1	69.72	1.43e-02	606.9	96.61	1.03e-02
5	547.8	87.19	1.14e-02	738.4	117.52	8.51e-03
6	547.8	87.19	1.14e-02	741.0	117.93	8.48e-03
7	715.7	113.91	8.78e-03	930.3	148.06	6.75e-03
8	715.7	113.91	8.78e-03	931.9	148.32	6.74e-03
9	937.3	149.17	6.70e-03	1188.9	189.22	5.28e-03
10	937.3	149.17	6.70e-03	1189.8	189.36	5.28e-03

The finite element mesh (see Fig. 1) includes 100 rectangular elements (100 random design variables), and the total number of degrees of freedom is 606 to simply supported plate (489 to clamped plate). One diagram show symmetry both conditions. Deterministic input data are: length $L = 6$ m, Young modulus E= 30GPa (concrete 25/30), Poisson ratio $\nu = 0.2$, load $F = 50$ kN, mass density $\rho = 0.24$.

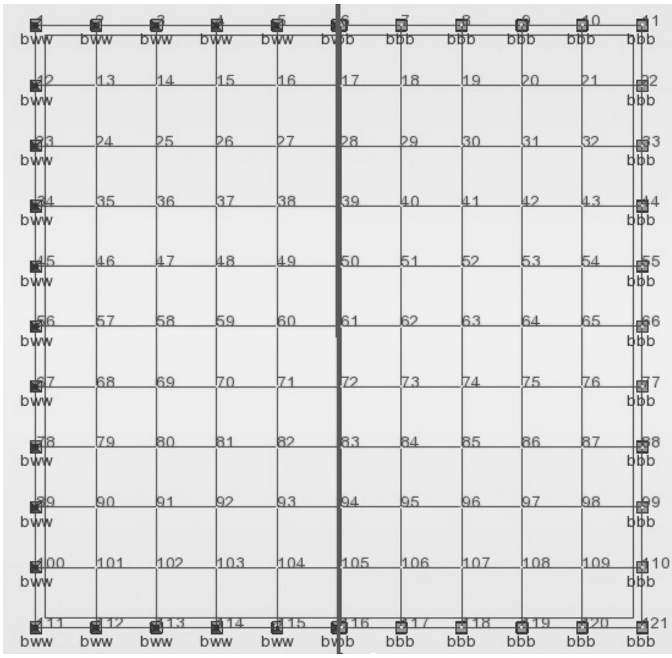

Fig. 1: 100-element plate. Boundary condition (left) for simply supported, (right) clamped plate. Boundary conditions means: bww – block Uz, free Ux, free Ry, bbb – block Uz, Ux and Ry

To solve the initial-terminal problem the mode superposition technique is used with 10 lowest eigensystems (see Tab. 1). The equations are integrated with respect to time for 1500 time steps (time interval $\Delta t = 0.002$). The time response of expectations and standard deviations of z-displacement at mid-point (compared against the deterministic solution) are displayed in Fig. 2a, Fig. 2b. and Fig. 3.

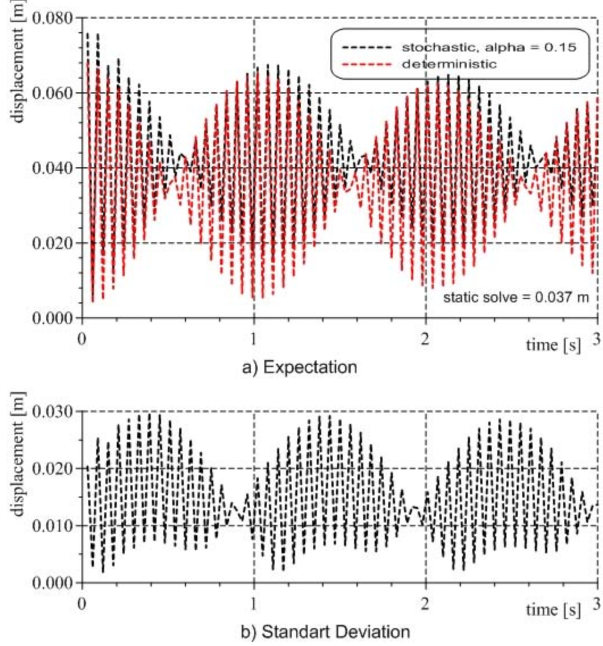

Fig. 2a: 100-element plate. Time response of stochastic displacement for simply supported plate

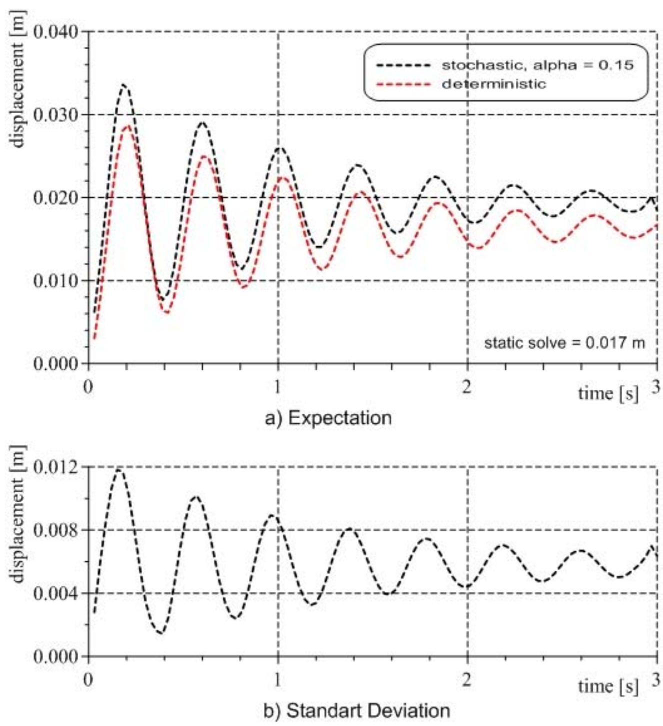

Fig. 2b: 100-element plate. Time response of stochastic displacement for clamped plate

The expectation of the thickness $E[t(x,y)] = 0.06$ m and the homogeneous to autocovariance function of the thickness is assumed to be

$$Cov\left[t\left(x_r, y_r\right), t\left(x_s, y_s\right)\right] = a^2 exp\left(\frac{x_r - x_s}{d_x}\right) exp\left(\frac{y_r - y_s}{d_y}\right)$$

where the standard deviation $\alpha = 0.15$, d_x and d_y are decay factors in the x and y direction and determined from the end-point correlation $R[t(x_0, y), t(x_1, y)]$ and $R[t(x, y_0), t(x, y_1)]$. In each problem the thickness at various points are perfectly correlated and the deflection at the center of the plates are calculated.

4 CONCLUSION

In the stochastic perturbation analysis we deal with one system of the zeroth-order equations, one system of the first-order equations for each of the random variables and one system of the second-order equations. This non-statistical approach does not restrict the analysis to some limits of random fields as in the statistical techniques; it is applicable to both the homogeneous and nonhomogeneous random fields and a normal approximation is not necessarily needed. The restriction of small uncertainties in random variables (about 15 %), being inherent of the mean-point perturbation procedure, is seemingly eliminated by the check-point perturbation scheme in which the point of the system is perturbated around its parameterized variables.

With the transformation from correlated random variables to uncorrelated variables and by using only dominant part of the transformed set, the algorithms worked out are effective even for PC-based stochastic analysis of large-scale systems with acceptable computations cost. Since almost all operations related to random quantities can be out by the procedures for deterministic calculations the algorithms developed can be immediately adapted to existing deterministic finite element programs.

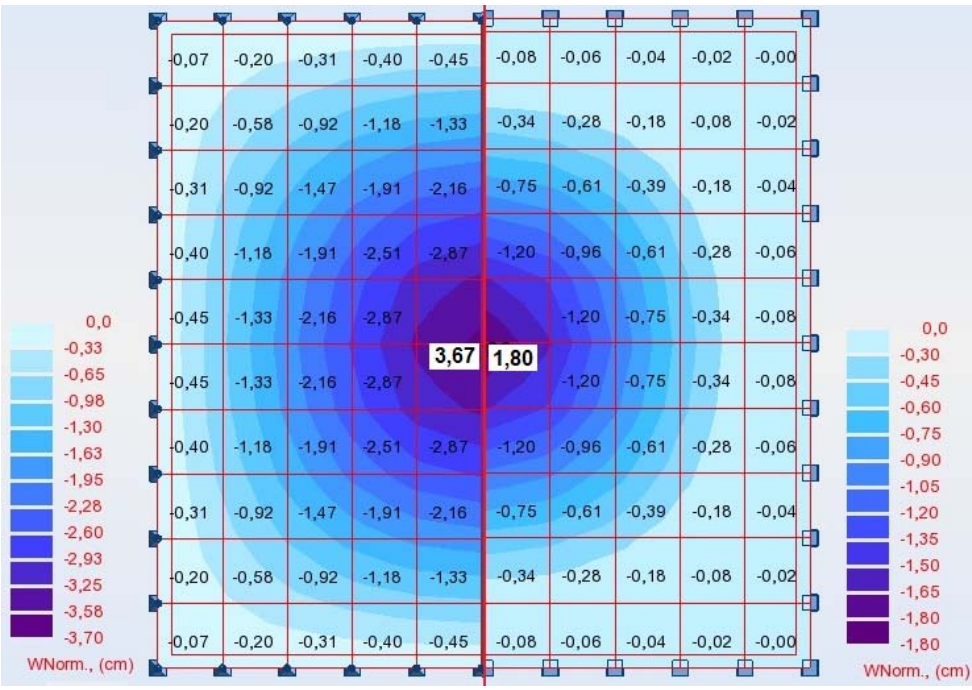

Fig. 3: 100-element plate. Deterministic displacement (left) for simply supported, (right) clamped plate

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