Ryszard WALENTYŃSKI[1]

GENERALIZED FUNCTIONS AND CALCULUS OPERATORS OF *MATHEMATICA* APPLIED TO EVALUATION OF INFLUENCE LINES AND ENVELOPES OF STATICALLY INDETERMINATE BEAMS

Abstract

The paper presents an analytical method of finding functions of influence lines of statically indeterminate beams. There are presented solutions of a fourth order equation with a right hand side with second and third derivative of Dirac delta. There is shown that their solution are influence lines of moments and transverse forces. Moreover, thanks to *Mathematica*, analytical form of envelopes functions can be evaluated.

Keywords

Influence lines, fundamental solution, generalized functions, Dirac delta, Heaviside step function, structural mechanics, differential equations, envelopes, *Mathematica*.

1 INTRODUCTION

Influence lines play an important role in education of structural engineers [1] and engineering practice, especially designing of bridges [2]. They are functions and called by mathematicians Green functions or fundamental solution and have important application in many engineering fields, see for example [3] and [4].

The purpose of this article is to show that by *Mathematica* [5] and built in it generalized functions and calculus operators it is possible to propose a new analytical approach for determining the influence lines and envelopes of internal forces in beams, especially statically indeterminate.

1.1 Generalized functions

We will use two generalized functions: Heaviside step function $\theta(x)$ and Dirac $\delta(x)$. They are implemented in *Mathematica* as **HeavisideTheta[x]** and **DiracDelta[x]**, respectively. Heaviside step function can be defined with the following piecewise function [6]

$$\theta(x) := \begin{cases} 0 & x < 0 \\ \dfrac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} . \tag{1}$$

Its derivative is a Dirac delta [7].

$$\theta'(x) = \delta(x) . \tag{2}$$

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[1] Prof. (assoc.) Ryszard Walentyński M.Sc. C.Eng. Ph.D. D.Sc., Chair of Building Structures Theory, Faculty of Civil Engineering, Silesian University of Technology, ul. Akademicka 5, 44-100 Gliwice, Poland, phone: (+48) 32 237 21 18, e-mail: Ryszard.Walentynski@polsl.pl.

Dirac delta can be defined with the following piecewise function

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}.$$ (3)

Its integral within any integral containing point $x = 0$ is equal to 1.

$$\int_{-\infty}^{\infty} \delta(x)\, dx = \int_{-a}^{a} \delta(x)\, dx = 1.$$ (4)

Both functions can be differentiate and integrate within *Mathematica*. More information can be found in [6] and [7].

1.2 Application of generalized functions in structural mechanics

Let us analyze a simply supported beam presented in Fig. 1.
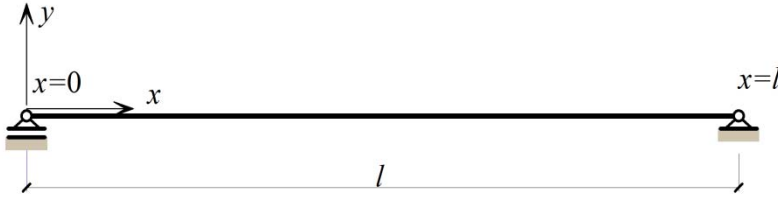
Fig. 1: Simply supported beam

Deflection function $y(x)$ of this beam loaded with a point load P in a distance α from the left support and with bending stiffness EJ can be evaluated by the solution of the following differential equation.

$$EJ\, y^{(4)}(x) = -P\, \delta(x - \alpha).$$ (5)

The following boundary conditions holds in the analyzed case.

$$y(0) = 0,\ y(l) = 0,\ y''(0) = 0,\ y''(l) = 0.$$ (6)

Mathematica has implemented operator for solving analytically differential equations **DSolve[]**. After evaluation the solution it is simplified with **FullSimplify[]** function, taking into account limitations of x and α (**Assuming[]**).

Assuming[$0 < \alpha < l$&&$0 < x < l$, *FullSimplify*[*DSolve*[{*EJ* y(4)[x] = = −P DiracDelta[x − α], y[0] == 0, y[l] == 0, y''[0] == 0, y''[l] = = 0}, y[x], x]]]

$\{\{y[x] \rightarrow \dfrac{1}{6\mathrm{EJ}l}(Px(l - \alpha)(x^2 + \alpha(-2l + \alpha)) - lP(x - \alpha)^3\,\mathrm{HeavisideTheta}[x - \alpha])\}\}$

Diagram of the deflection of this beam when the force is placed in the middle of it is presented in Fig. 2.
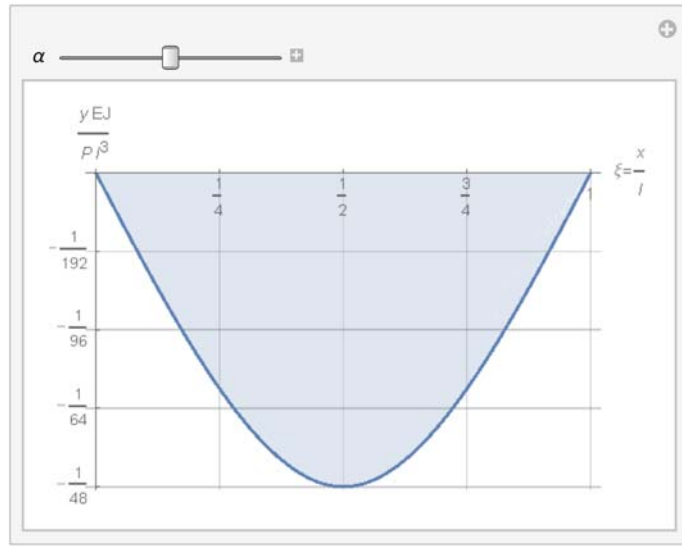
Fig. 2: Deflection of simply supported beam (Fig. 1) loaded with a point load P
in the middle of span

When the beam is loaded with a point moment (pair of forces) K the differential equation is defined with, where position of the moment is defined with the first derivative of the Dirac delta.

$$EJ\ y^{(4)}(x) = K\ \delta'(x-\alpha)\ . \tag{7}$$

Assuming$[0 < \alpha < l \&\& 0 < x < l,$ **FullSimplify**$[$**DSolve**$[\{$**EJ**$y^{(4)}[x] = = K$**DiracDelta**$'[x - \alpha], y[0] == 0, y[l] == 0, y''[0] == 0, y''[l] = = 0\}, y[x], x]]]$

$\{\{y[x] \to \dfrac{1}{6EJl} K(-x(2l^2 + x^2 - 6l\alpha + 3\alpha^2) + 3l(x - \alpha)^2 \text{HeavisideTheta}[x - \alpha])\}\}$

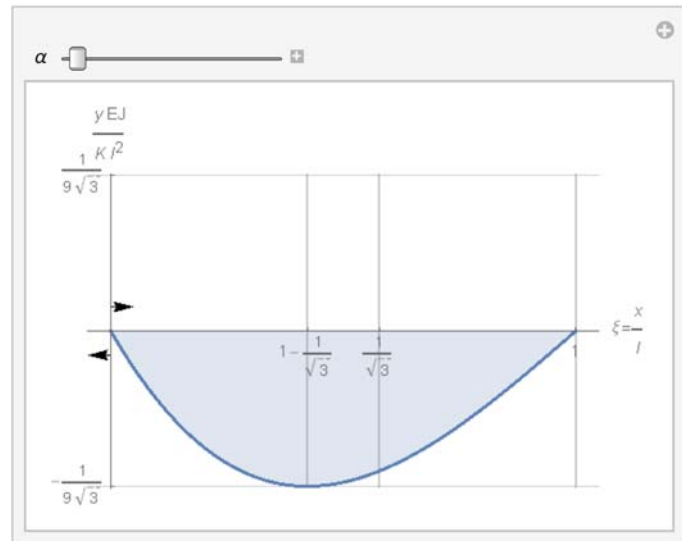Deflection of the beam loaded with a pair of forces on the left support is shown in Fig. 3.

Fig. 3: Deflection of simply supported beam (Fig. 1) loaded with a point moment K
in the left support

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1.3 Computational experiment – motivation of research

Doing computational experiment with higher derivatives of Dirac delta:

$$y^{(4)}(x) = \delta''(x-a), \tag{8}$$

$$y^{(4)}(x) = \delta^{(3)}(x-a), \tag{9}$$

it has been found that the obtained functions looked like influence lines of simply supported beam [1] for moments and transverse forces, respectively. It is a main motivation for the presented analyses, since it can be a chance to find more general analytic solution of the problem.

To understand how it works we will start with analysis of simple supported beam showing that eqns (8) and (9) can be derived from the influence lines. Next we will show that influence lines can be derived from these equations for statically indeterminate structures. There is presented that envelopes of the functions can be derived analytically.

2 ANALYSIS OF SIMPLE SUPPORTED BEAM

2.1 Dimensionless coordinates

First we introduce a dimensionless coordinates measured along the beam. They will start in the middle of the beam.
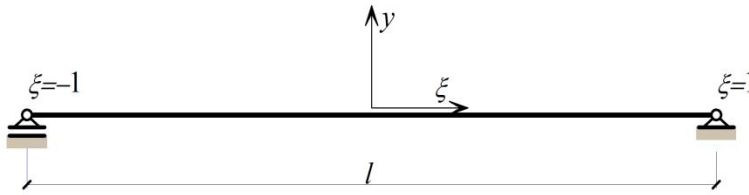

Fig. 4: Hinged-hinged (simply supported) beam with dimensionless coordinates

The physical coordinates shown in Fig. 1 are related with dimensionless with the following formula.

$$x := \frac{(1+\xi)l}{2}. \tag{10}$$

2.2 Influence line for transverse force

When a considered cross-section has coordinate α the influence line of transverse force can be described with the following piecewise function [1]

$$IL\,T(\xi,\alpha) := -\frac{1}{2}\begin{cases}\xi+1 & \alpha < \xi \\ \xi-1 & \alpha > \xi\end{cases}. \tag{11}$$

It can be rewritten using Heaviside step function:

$$IL\,T(\xi,\alpha) := \theta(\xi-\alpha) - \frac{1+\xi}{2}. \tag{12}$$

Differentiating both sides of (12) four times with regard of ξ we obtain the following differential equation (compare (9)):

$$IL\,T^{(4,0)}(\xi,\alpha) = \delta^{(3,0)}(\xi-\alpha). \tag{13}$$

Since we will analyze the equation (13) with regard of the first variable, we can set $IL\,T\,(\xi,\alpha)\to y(\xi)$ and boundary conditions

$$y(-1)=0,\ y(1)=0,\ y''(-1)=0,\ y''(1)=0\ . \tag{14}$$

Solution with Mathematica is done with, where solution of differential equation with its boundary conditions (**Dsolve**) is simplified (**FullSimplify**) taking into account domain of the variables ξ and α (**Assuming**).

Assuming$[-1<\alpha<1\&\&-1<\xi<1,$

FullSimplify[

DSolve$[\{y^{(4)}[\xi]==\textbf{DiracDelta}^{(3)}[\xi-\alpha],y[-1]==0,y[1]==0,y''[-1]==0,$

$y''[1]==0\},y[\xi],\xi]]]$

returns expresion (12)

$$\{\{y[\xi]\to-\frac{1}{2}-\frac{\xi}{2}+\text{HeavisideTheta}[-\alpha+\xi]\}\}$$

what proves that the method works. The method has a physical sense[2].

Diagrams of the influence lines for 3 different values of α are presented in Figs. 5, 6 and 7.
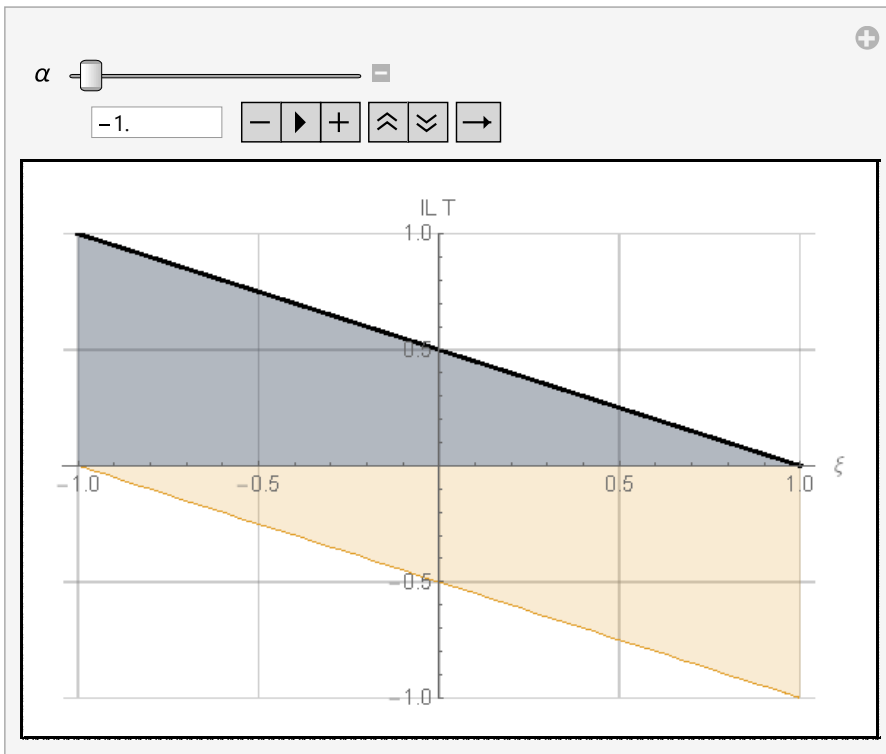
Fig. 5: Influence line of transverse force in hinged-hinged beam for $\alpha=-1$

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[2] Author is an engineer and therefore presents the problem from his practical point of view. Formal mathematical proof and interpretation should be done by a mathematician.
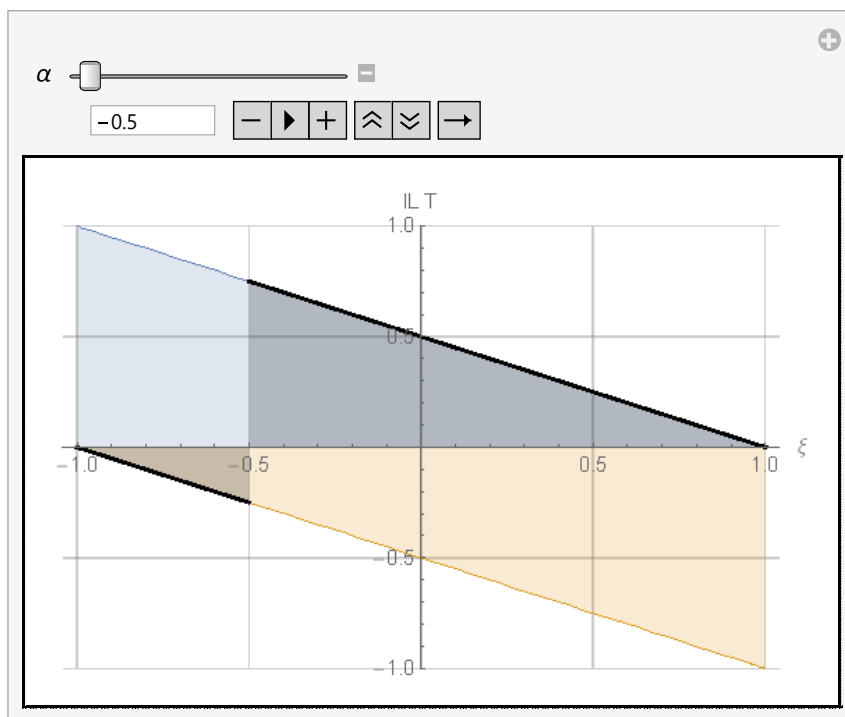
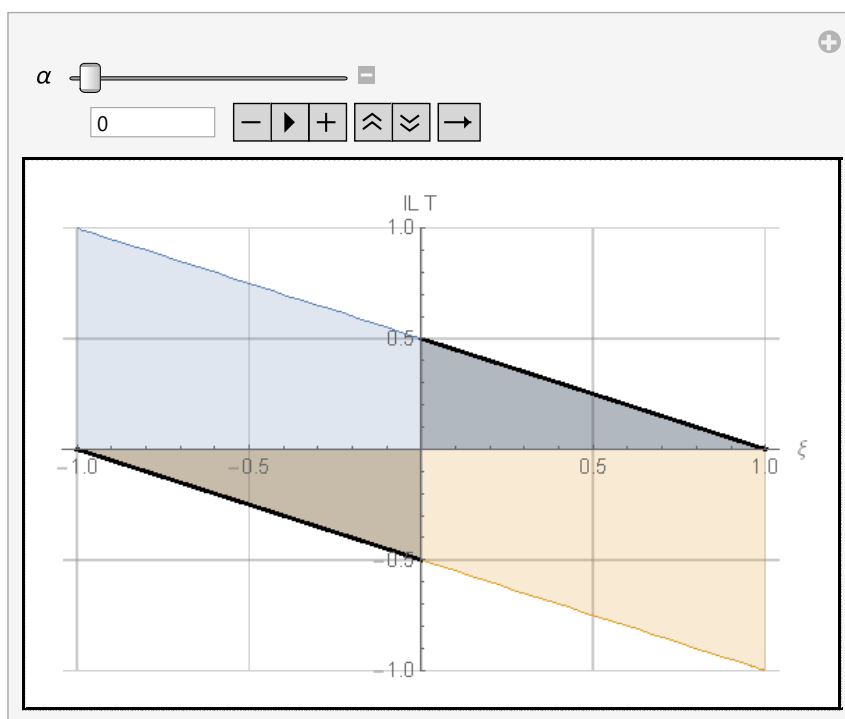Fig. 6: Influence line of transverse force in hinged-hinged beam for $\alpha = -\dfrac{1}{2}$

Fig. 7: Influence line of transverse force in hinged-hinged beam for $\alpha = 0$

2.3 Influence line for bending moment

Similar analysis can be done for bending moments and it may be shown, using similar analysis like in the previous subsestion, that differential equation describing the problem takes the form:

$$IL\, M^{(4,0)}(\xi,\alpha) = -\frac{1}{2}\delta^{(2,0)}(\xi - \alpha)\, l\,. \tag{15}$$

Setting similarly like in the previous subsection $IL\, M(\xi,\alpha) \rightarrow y(\xi)$ we solve

Assuming[$-1 < \alpha < 1$&& $-1 < \xi < 1$, FullSimplify[DSolve[{$y^{(4)}[\xi] =$
$$= -\frac{1}{2}\textbf{DiracDelta''}[\xi - \alpha]l, y[-1] == 0, y[1] == 0, y''[-1] == 0, y''[1] =$$
$$= 0\}, y[\xi], \xi]]]$$

$$\{\{y[\xi] \rightarrow \frac{1}{4}l(-(-1 + \alpha)(1 + \xi) + 2(\alpha - \xi)\text{HeavisideTheta}[-\alpha + \xi])\}\}$$

Piecewise form is well known influence line of moments in a simply supprted beam, compare [1].

$$IL\, M(\xi,\alpha) := \frac{1}{4}\begin{cases}(1-\alpha)(1+\xi) & \alpha > \xi \\ (1+\alpha)(1-\xi) & \alpha < \xi\end{cases}. \tag{16}$$

Diagrams of influence lines for 2 different values of α are presented in Figs.8 and 9.
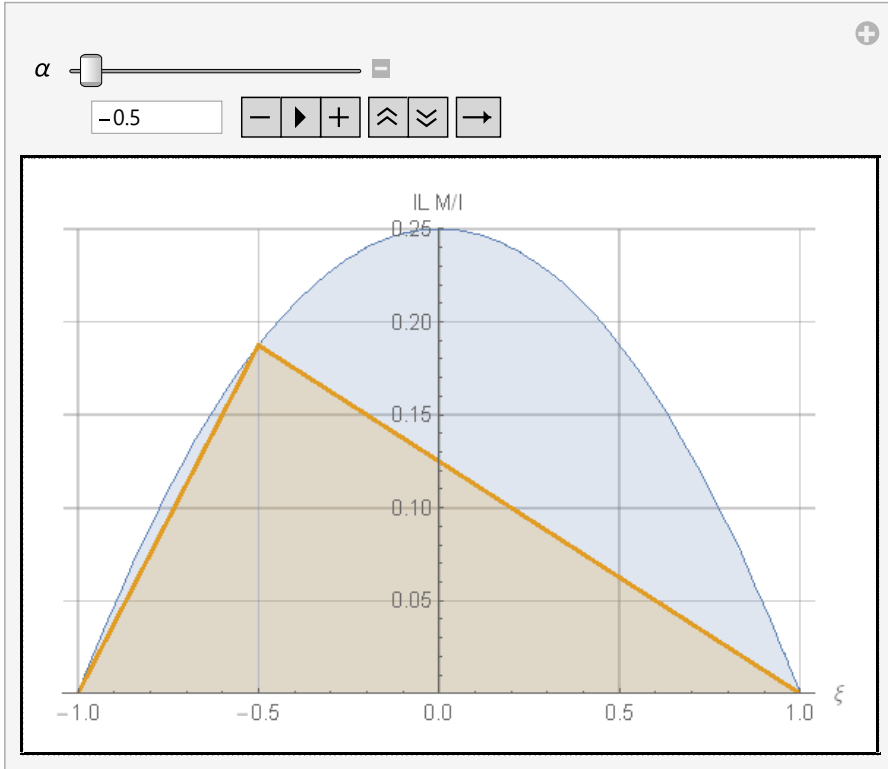

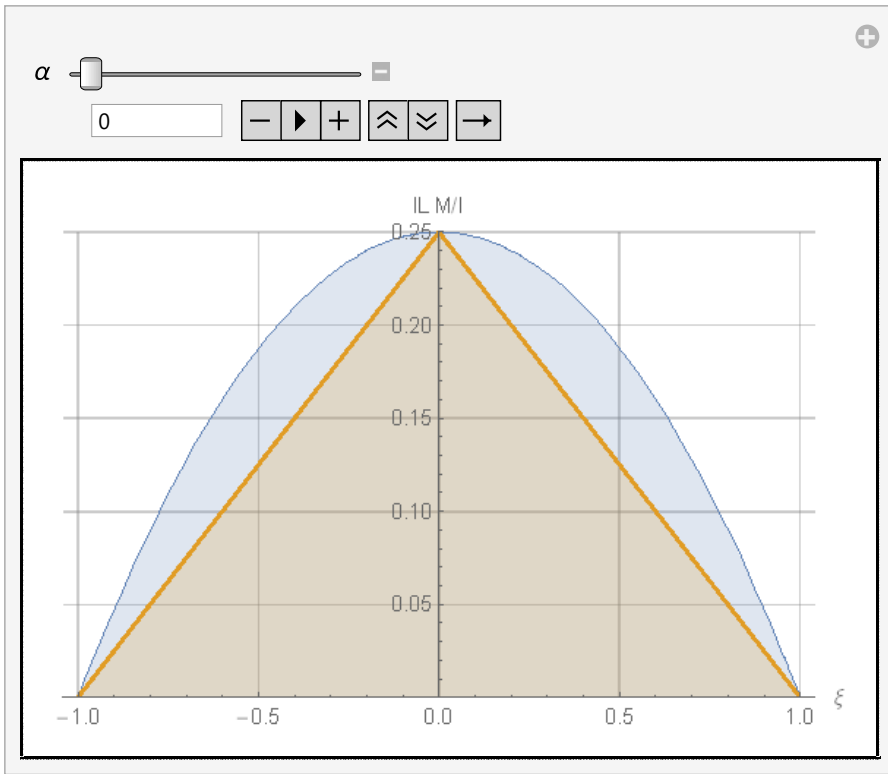Fig. 8: Influence line of bending moment in hinged-hinged beam for $\alpha = -\frac{1}{2}$

Fig. 9: Influence line of bending moment in hinged-hinged beam for $\alpha = 0$

3 ANALYSIS OF CLAMPED-CLAMPED BEAM

Now we can show that the method work also in case of statically indeterminate structure, for example a clamped-clamped beam, shown in Fig. 10.
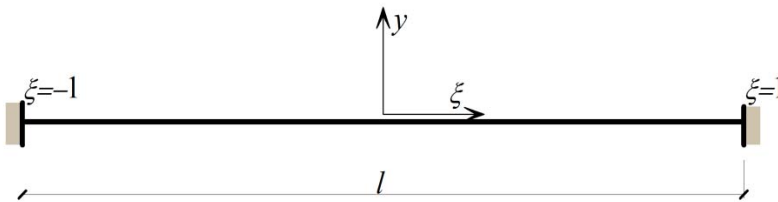

Fig. 10: Clamped-clamped beam with dimensionless coordinates

To analyse the problem it enough to change boundary conditions (14) to

$$y(-1) = 0, \; y(1) = 0, \; y'(-1) = 0, \; y'(1) = 0. \tag{17}$$

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3.1 Influence line for bending moment

The solution of differential equation (15) with boundary conditions (17) leads to the following result:

Assuming$[-1 < \alpha < 1 \&\& -1 < \xi < 1,$ **FullSimplify**$[$**DSolve**$[\{y^{(4)}[\xi] =$
$$= \frac{1}{2}\textbf{DiracDelta}''[\xi - \alpha]l, y[-1] == 0, y[1] == 0, y'[-1] == 0, y'[1] =$$
$$= 0\}, y[\xi], \xi]]]$$

$$\left\{\left\{y[\xi] \to \frac{1}{8}l(-(1 + \alpha(-2 + \xi))(1 + \xi)^2 + 4(-\alpha + \xi)\text{HeavisideTheta}[-\alpha + \xi])\right\}\right\}$$

what can be expressed in piecewise form.

$$IL\,M(\xi,\alpha) := -\frac{l}{8}\begin{cases}(1+\alpha(\xi-2))(1+\xi)^2 & \alpha > \xi \\ (1+\alpha(\xi+2))(1-\xi)^2 & \alpha < \xi\end{cases}. \tag{18}$$

Figures 11-14 shows 4 influence lines for different position of cross-sections on background of their envelopes.
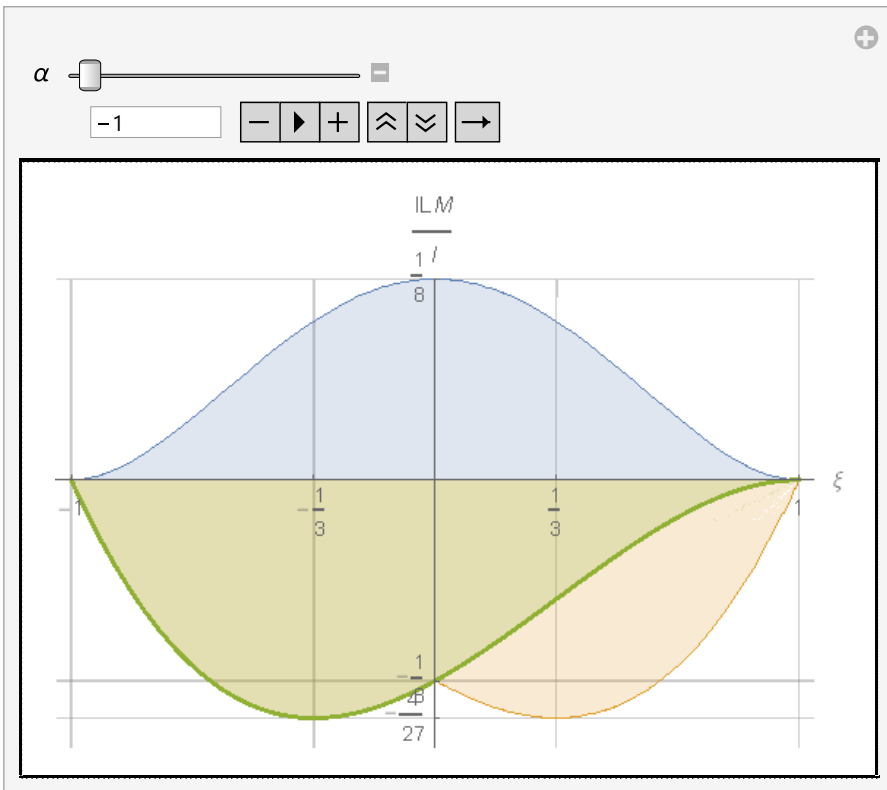

Fig. 11: Influence line of bending moment in clamped-clamped beam for $\alpha = -1$
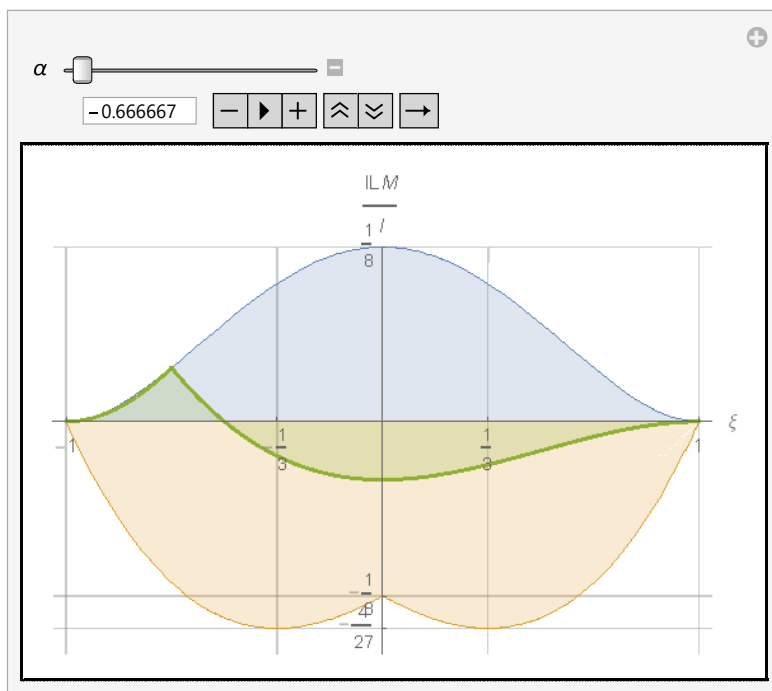
Fig. 12: Influence line of bending moment in clamped-clamped beam for $\alpha = -\dfrac{1}{6}$
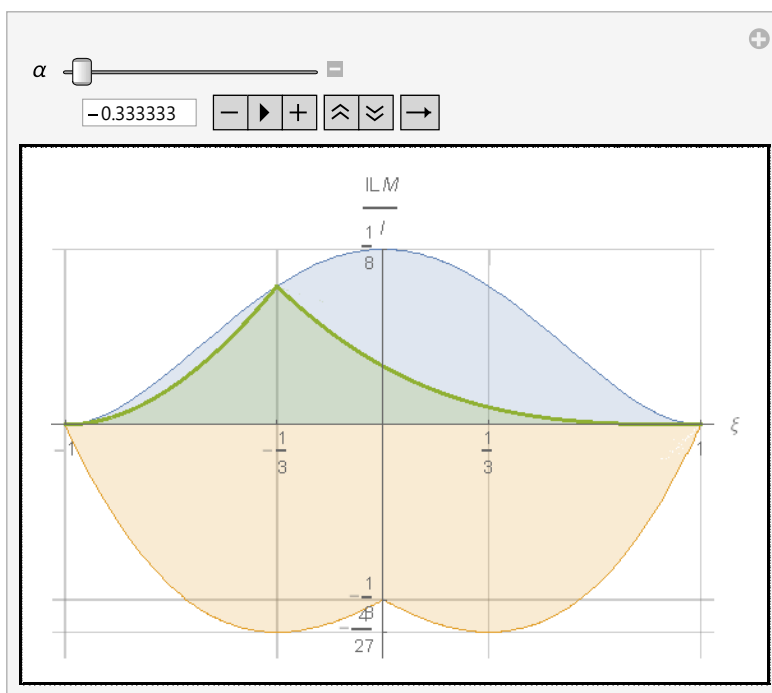

Fig. 13: Influence line of bending moment in clamped-clamped beam for $\alpha = -\dfrac{1}{3}$
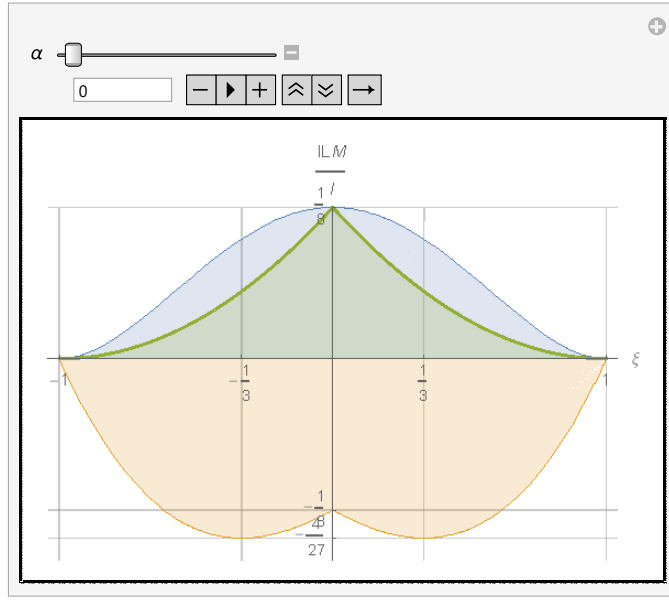
Fig. 14: Influence line of bending moment in clamped-clamped beam for $\alpha = 0$

It is worth to mention that for $\alpha \in \left(-1, -\dfrac{1}{3}\right)$ and $\alpha \in \left(\dfrac{1}{3}, 1\right)$ the function of influence line changes the sign and therefore for cross-section in these intervals maximum and minimum moments from live loads are not produced by the loads distributed all over the span. Moreover the minimal moments in supports caused by moving point load are extremal when the force is placed in 1/3 of the span: $M(-1, -\dfrac{1}{3}) = -\dfrac{4}{27} P\, l \approx -0.1481 P\, l$ and its absolute value is 18.5% bigger of the one computed when the force is placed in the middle of the span $M(-1, 0) = -\dfrac{1}{8} P\, l = -0.125 P\, l$.

3.2 Influence line for tranverse force

The solution of of differeantial equation (13) with boundary conditions (17) leads to the following result:

$\textbf{Assuming}[-\textbf{1} < \alpha < \textbf{1} \&\& -\textbf{1} < \xi < \textbf{1}, \textbf{FullSimplify}[\textbf{DSolve}[\{y^{(4)}[\xi] =$
$= \textbf{DiracDelta}^{(3)}[\xi - \alpha], y[-\textbf{1}] == \textbf{0}, y[\textbf{1}] == \textbf{0}, y'[-\textbf{1}] == \textbf{0}, y'[\textbf{1}] =$
$= \textbf{0}\}, y[\xi], \xi]]]$

$$\{\{y[\xi] \to \frac{1}{4}(-2 + \xi)(1 + \xi)^2 + \text{HeavisideTheta}[-\alpha + \xi])\}\}$$

So the solution can be presented in generalized form:

$$IL\, T(\xi, \alpha) := \theta(\xi - \alpha) + \frac{1}{4}(\xi - 2)(1 + \xi)^2, \tag{19}$$

or piecewise one

$$IL\, T(\xi, \alpha) := \frac{1}{4}\begin{cases}(\xi - 2)(1 + \xi)^2 & \alpha > \xi \\ (\xi + 2)(1 - \xi)^2 & \alpha < \xi\end{cases}. \tag{20}$$

Figures 15-17 shows 3 influence lines for different position of cross-sections on background of their envelopes.
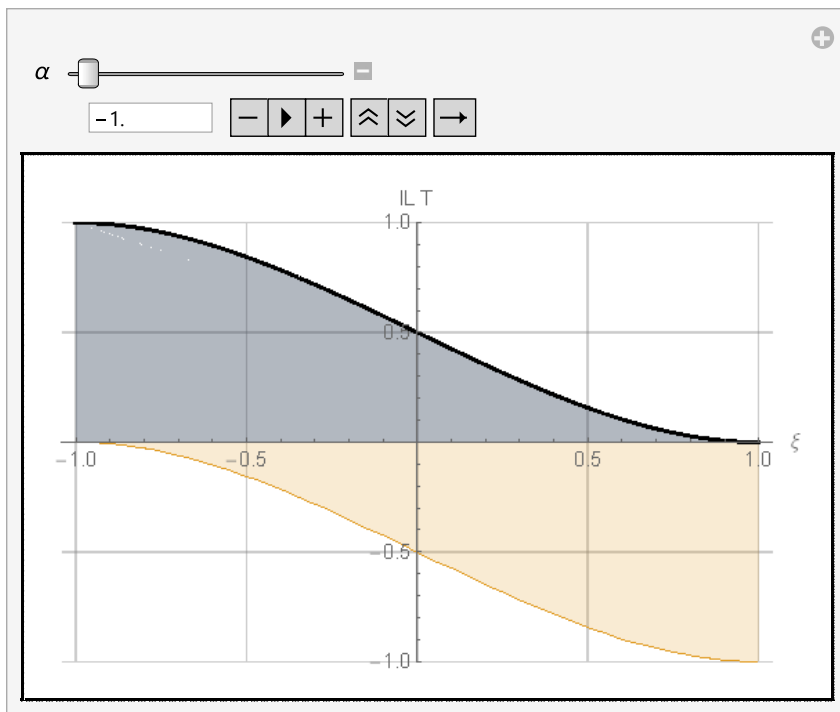
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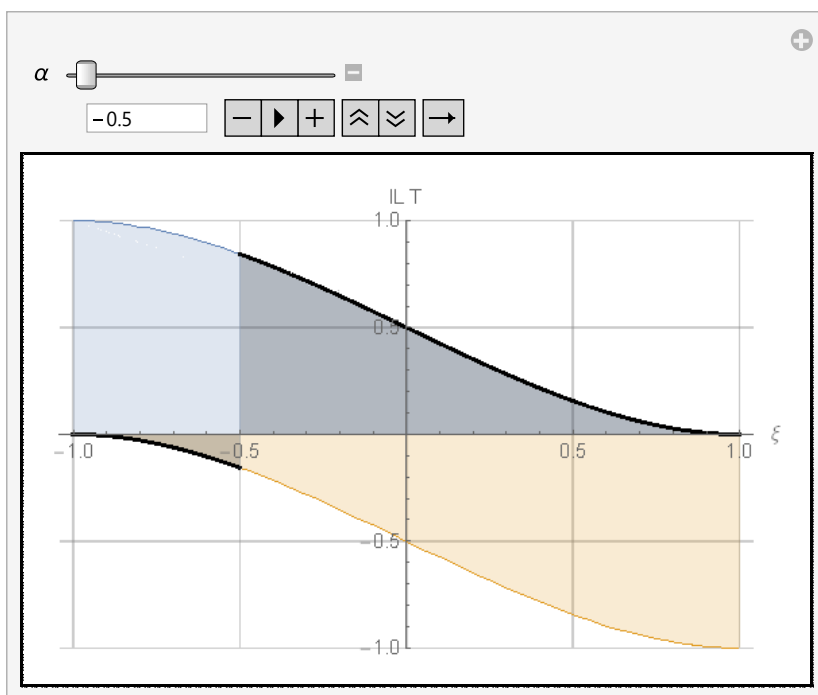Fig. 15: Influence line of transverse force in clamped-clamped beam for $\alpha = -1$

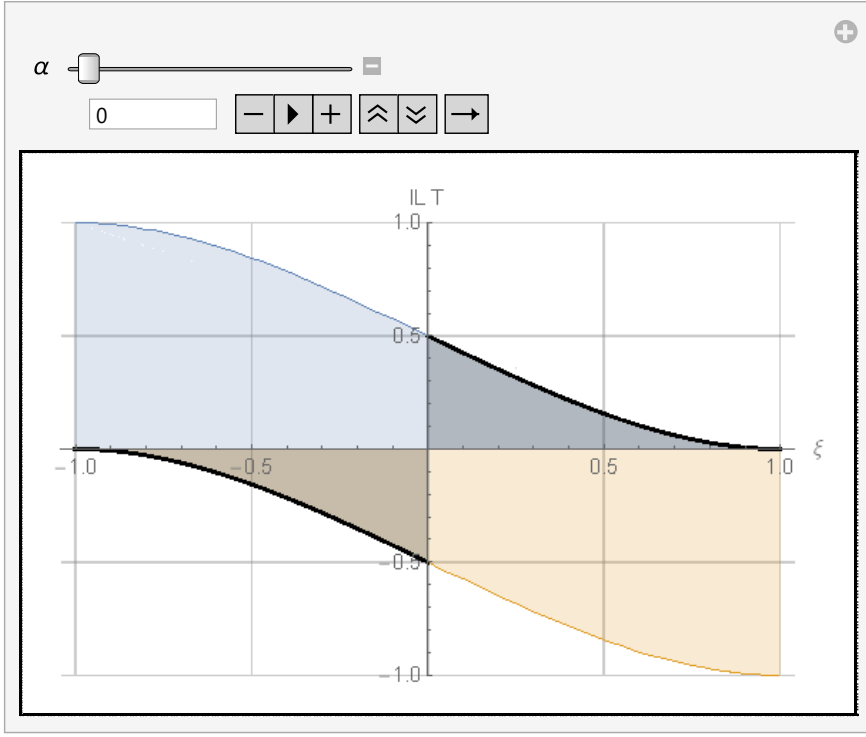Fig. 16: Influence line of transverse force in clamped-clamped beam for $\alpha = -\dfrac{1}{2}$

Fig. 17: Influence line of transverse force in clamped-clamped beam for $\alpha = 0$

4 ENVELOPES FOR UNIFORMLY DISTRIBUTED LOADS

4.1 General formulas

Let us consider that the beam is loaded with dead load g, which is distributed all over the domain and life load p, which can be distributed anywhere. In points where both loads are present intensity of load is:

$$q = g + p. \tag{21}$$

Influence lines are functions and for dead load we can compute values of dead load envelope by the following integral

$$S_g\left(\alpha\right) := g\int_{-1}^{1} IL\ S\left(\xi,\alpha\right) \mathrm{d}\xi, \tag{22}$$

where S is in the considered case bending moment M or transverse force T.

For the life load we have to consider for each cross-section adverse distribution of load. It can be done automatically within Mathematica by filtering positive and negative values with Heaviside step function of influence lines with the following integrals.

$$S_{p+}\left(\alpha\right) := p\int_{-1}^{1} IL\ S\left(\xi,\alpha\right) \theta\left(IL\ S\left(\xi,\alpha\right)\right) \mathrm{d}\xi, \tag{23}$$

$$S_{p-}\left(\alpha\right) := p\int_{-1}^{1} IL\ S\left(\xi,\alpha\right) \theta\left(-IL\ S\left(\xi,\alpha\right)\right) \mathrm{d}\xi. \tag{24}$$

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Having these function we can find envelopes by the following formulas:

$$S_{q+} := S_g + \frac{p}{q} S_{p+},\qquad (25)$$

$$S_{q-} := S_g + \frac{p}{q} S_{p-}.\qquad (26)$$

4.2 Envelopes of bending moment for clamped-clamped beam

Using scheme described by equations (22)-(24) we obtain analytic functions of envelopes of bending moments.

$$M_g(\xi) := \frac{g\, l^2 \left(1 - 3\xi^2\right)}{24},\qquad (27)$$

$$M_{p+}(\xi) := p\, l^2 \begin{cases} -\dfrac{(1+\xi)^3 \left(1 + 3\xi(3+8\xi)\right)}{192\,\xi^3} & -1 \le \xi < -\dfrac{1}{3} \\[2mm] \dfrac{\left(1-3\xi^2\right)}{24} & -\dfrac{1}{3} \le \xi \le \dfrac{1}{3} \\[2mm] \dfrac{(1-\xi)^3 \left(1 - 3\xi(3-8\xi)\right)}{192\,\xi^3} & \dfrac{1}{3} < \xi \le 1 \end{cases},\qquad (28)$$

$$M_{p-}(\xi) := p\, l^2 \begin{cases} \dfrac{(1+3\,\xi)^4}{192\,\xi^3} & -1 \le \xi \le -\dfrac{1}{3} \\[2mm] 0 & -\dfrac{1}{3} < \xi < \dfrac{1}{3} \\[2mm] -\dfrac{(1-3\,\xi)^4}{192\,\xi^3} & \dfrac{1}{3} \le \xi \le 1 \end{cases}.\qquad (29)$$

They are presented in Fig. 18. The function (27) is a inner parabola. It coincides with upper envelope (28) in the interval $\alpha \in \left\langle -\dfrac{1}{3}, \dfrac{1}{3} \right\rangle$. Lower function (29) is tangent to (27) in points $\xi = -1$ and $\xi = 1$. Functions (28) and (29) are piecewise, but their continuity in points $\xi = -\dfrac{1}{3}$ and $\xi = \dfrac{1}{3}$ have class C^3.

From the engineering point of view the important conclusion is that in points $\xi = -\dfrac{1}{\sqrt{3}}$ and $\xi = \dfrac{1}{\sqrt{3}}$ where function (27) have value 0 the envelopes are not null and if engineer designs here construction joint it must have some bearing capacity to take the bending moment caused by life load.
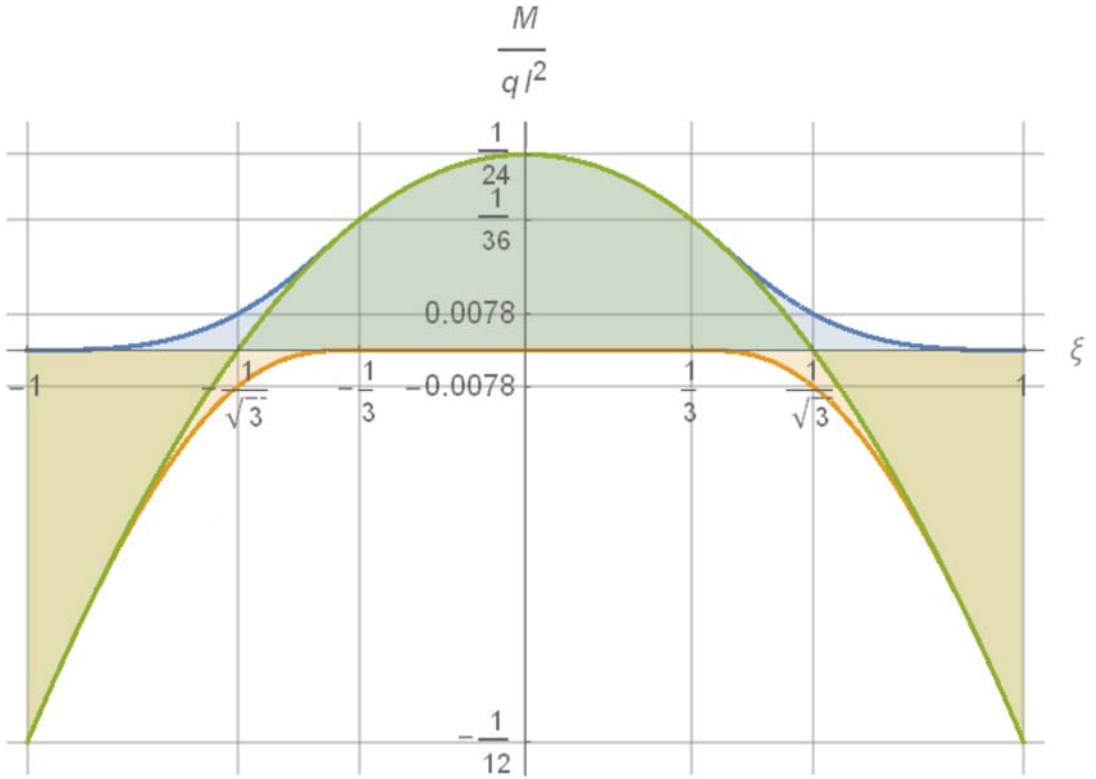
Fig. 18: $M_g(\xi)$, $M_{p+}(\xi)$, $M_{p-}(\xi)$ functions for clamped-clamped beam

Envelopes of transverse force for clamped-clamped beam

Applying scheme (22)-(24) to influence lines (19) we obtain

$$T_g := -\frac{g\,\xi}{2}, \tag{30}$$

$$T_{p-} := \frac{(\xi-3)(1+\xi)^3}{16}, \tag{31}$$

$$T_{p+} := \frac{(\xi+3)(1-\xi)^3}{16}. \tag{32}$$
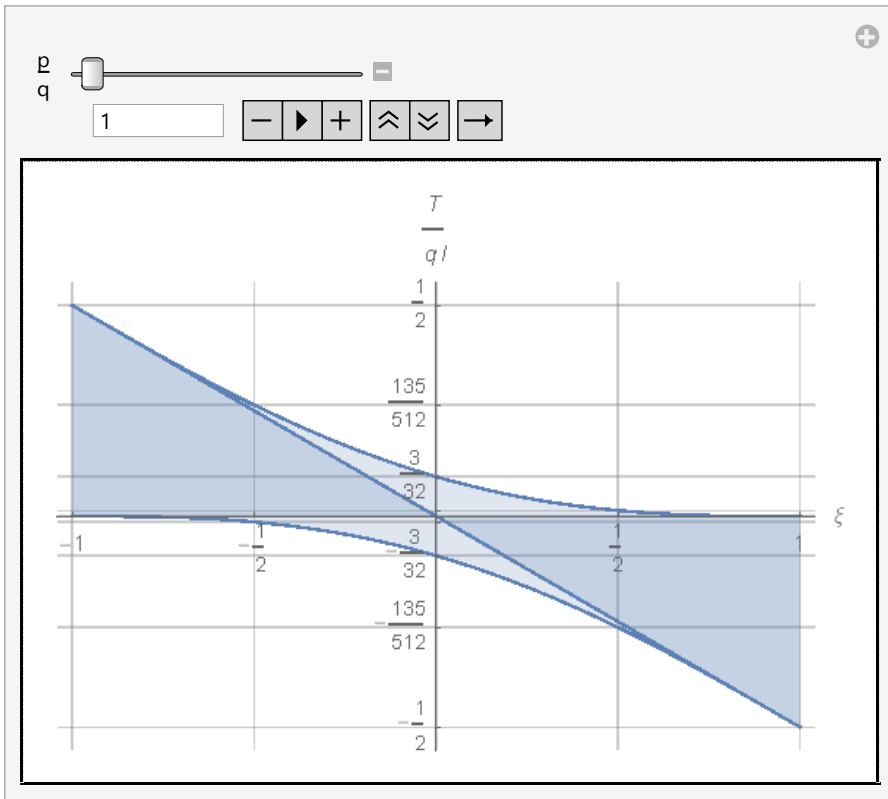
Their diagrams are presented in Fig. 19.

Fig. 19: $T_g(\xi)$, $T_{p+}(\xi)$, $T_{p-}(\xi)$ functions for clamped-clamped beam

5 TWO SPAN BEAM

5.1 Solution of differential equation

Finally we will analyze two span continuous beam simply supported for $\xi = -1$ and $\xi = 0$ and clamped for $\xi = 1$. Spans have length l and the scheme is shown in Fig. 20.
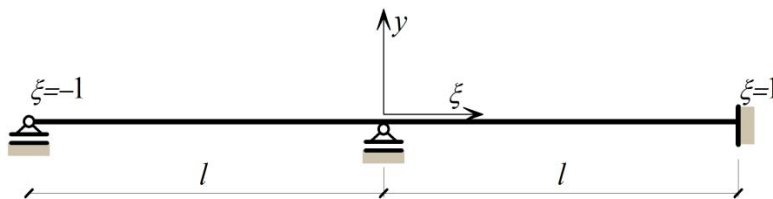
Fig. 20: Two span hinged-clamped beam with dimensionless coordinates

The problem is described with a system of two differential equations with regard of functions y (left hand span) and y_1 (right hand span) and boundary conditions presented below. In middle support vertical displacement is equal to zero for both functions and the first and second derivative of them are continuous. These requirements produce additional boundary conditions. In the presented below *Mathematica* expression by DSolve operator we have in the first curly brackets 2 differential equations followed by 8 boundary conditions. Next we have functions with regard which the system should be solved, and finally the variable.

Assuming[$-1 < \alpha < 1$&& $-1 < \xi < 1$, FullSimplify[DSolve[$\{y^{(4)}[\xi] =$
$$= \textbf{DiracDelta}''[\xi - \alpha]l, y1^{(4)}[\xi] == \textbf{DiracDelta}''[\xi - \alpha]l, y[-1] == 0, y1[0]$$
$$== 0, y[0] == 0, y'[0] == y1'[0], y''[0] == y1''[0], y1[1] == 0, y''[-1] =$$
$$= 0, y1'[1] == 0\}, \{y[\xi], y1[\xi]\}, \xi]]]$$

$$\{\{y[\xi] \to \frac{1}{7}(l(1 + \xi)((-2 + 3\alpha)\xi(2 + \xi) + \alpha(7 - 5\xi(2 + \xi))\text{HeavisideTheta}[-\alpha]) + 7l(-\alpha$$
$$+ \xi)\text{HeavisideTheta}[-\alpha + \xi]), y1[\xi]$$
$$\to \frac{1}{7}l(\xi(-4 + 3(-2 + \xi)\xi + \alpha(6 + (9 - 8\xi)\xi)) + \alpha(-1 + \xi)^2(7$$
$$+ 11\xi)\text{HeavisideTheta}[-\alpha] + 7(-\alpha + \xi)\text{HeavisideTheta}[-\alpha + \xi])\}\}$$

5.2 Influence lines of bending moment two span beam
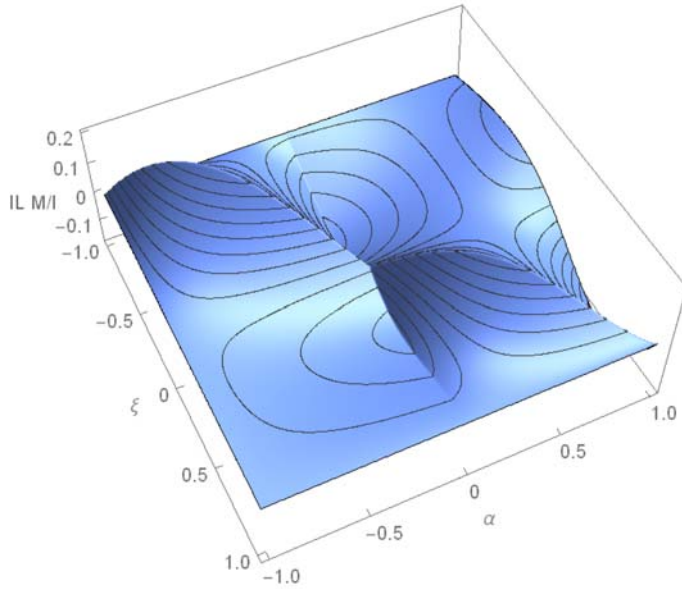
The solution is presented in Fig. 21.

Fig. 21: 3D diagram of influence function $IL\,M(\xi, \alpha)$ of two span hinged-clamped beam

Several influence lines on background of their envelopes is shown in Figs. 22-29.
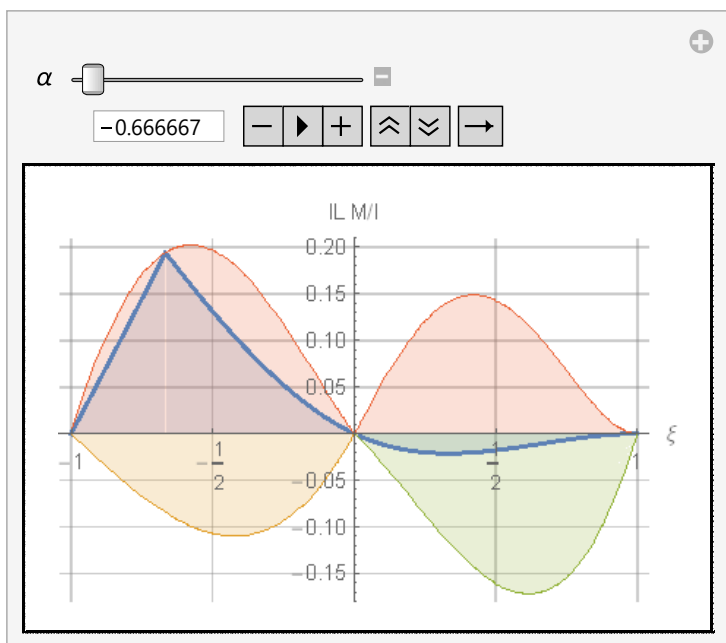
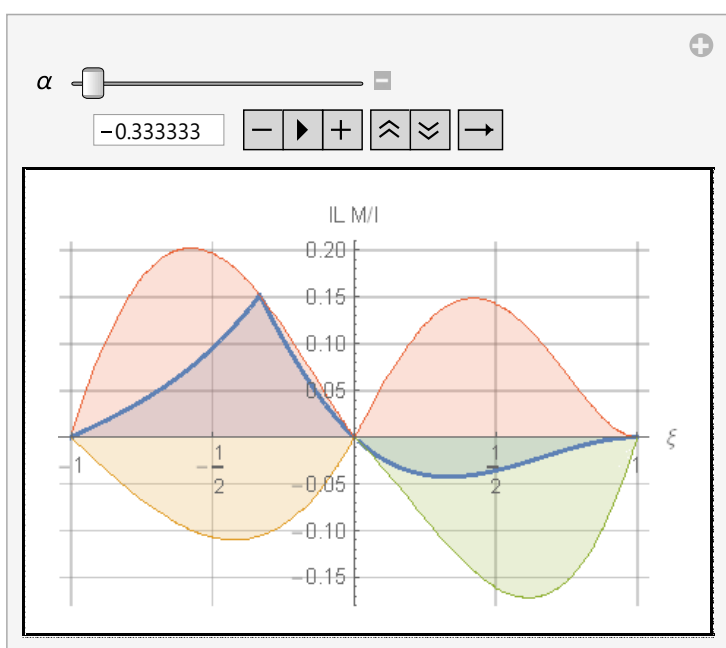Fig. 22: Influence function $IL\ M\left(\xi, -\dfrac{2}{3}\right)$ of two span hinged-clamped beam

Fig. 23: Influence function $IL\ M\left(\xi, -\dfrac{1}{3}\right)$ of two span hinged-clamped beam
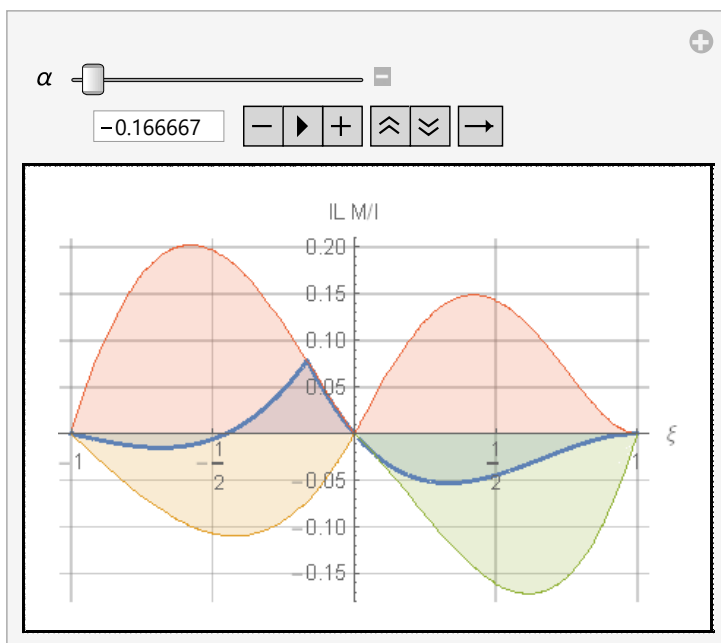
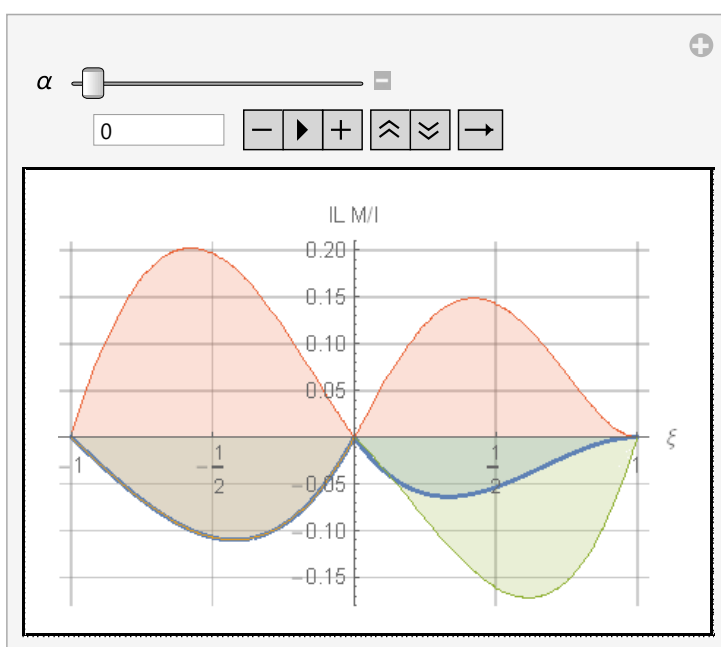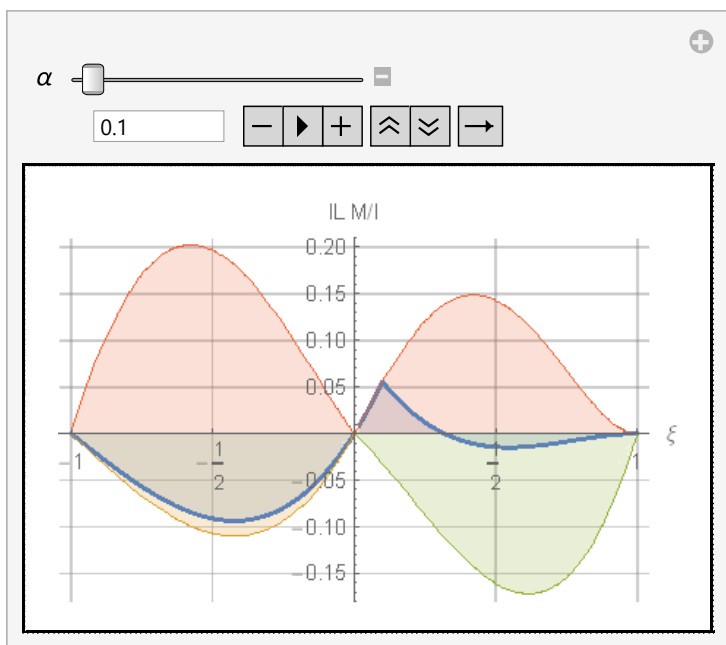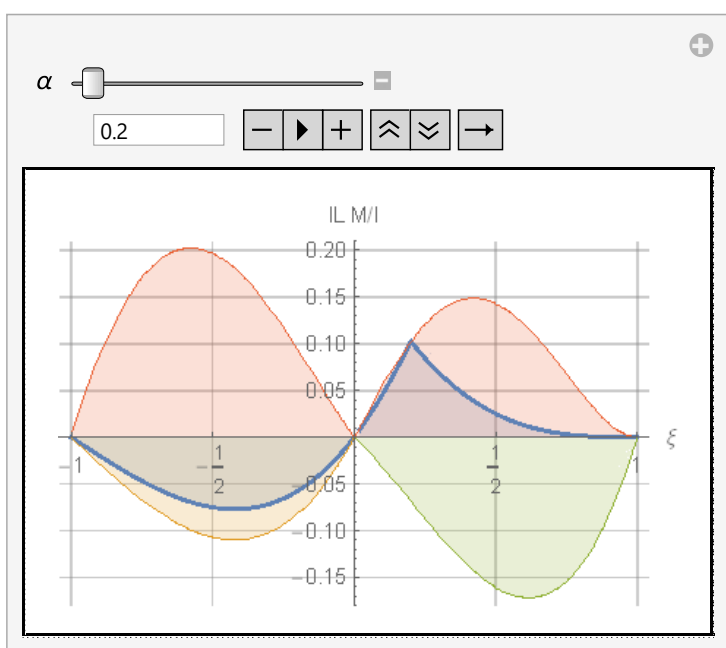Fig. 24: Influence function $IL\ M\left(\xi, -\dfrac{1}{6}\right)$ of two span hinged-clamped beam
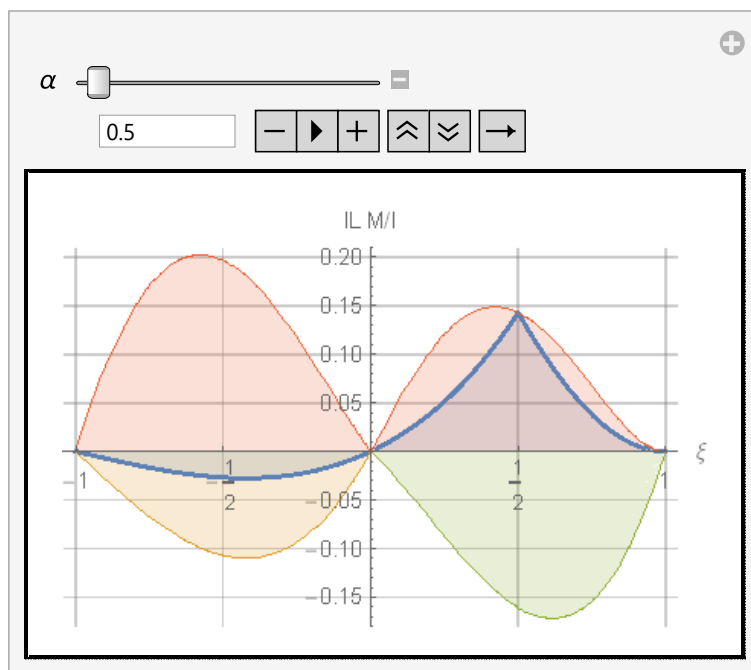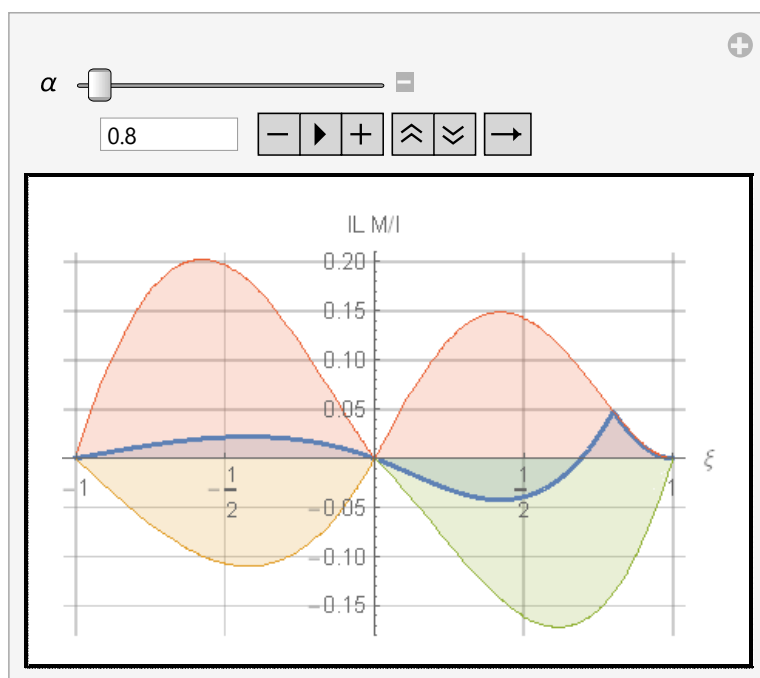

Fig. 25: Influence function $IL\ M\left(\xi, 0\right)$ of two span hinged-clamped beam

Fig. 26: Influence function $IL\ M\left(\xi, \dfrac{1}{10}\right)$ of two span hinged-clamped beam

Fig. 27: Influence function $IL\ M\left(\xi, \dfrac{1}{5}\right)$ of two span hinged-clamped beam

Fig. 28: Influence function $IL\ M\left(\xi, \dfrac{1}{2}\right)$ of two span hinged-clamped beam

Fig. 29: Influence function $IL\ M\left(\xi, -\dfrac{2}{3}\right)$ of two span hinged-clamped beam

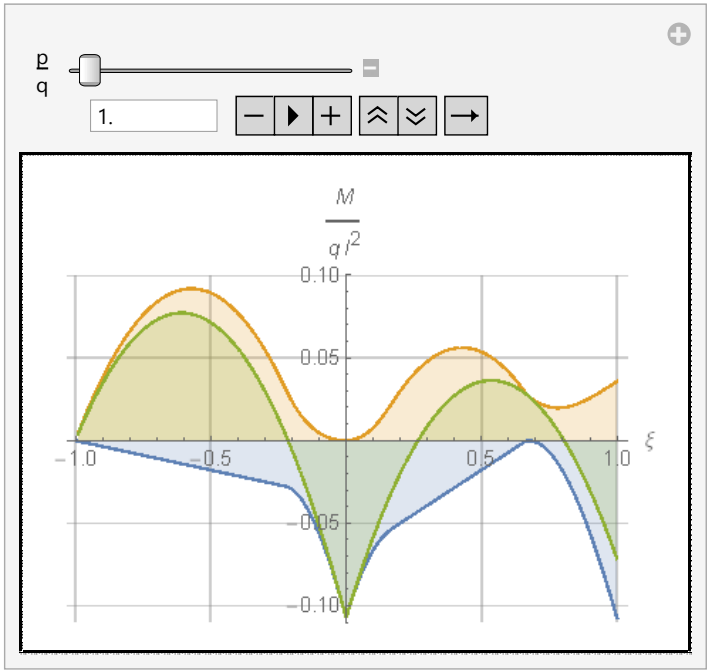Envelopes of bending moments for two different proportions of p/q are presented in Figs. 30 and 31.

Fig. 30: $M_g(\xi)$, $M_{p+}(\xi)$, $M_{p-}(\xi)$ functions of two span hinged-clamped beam
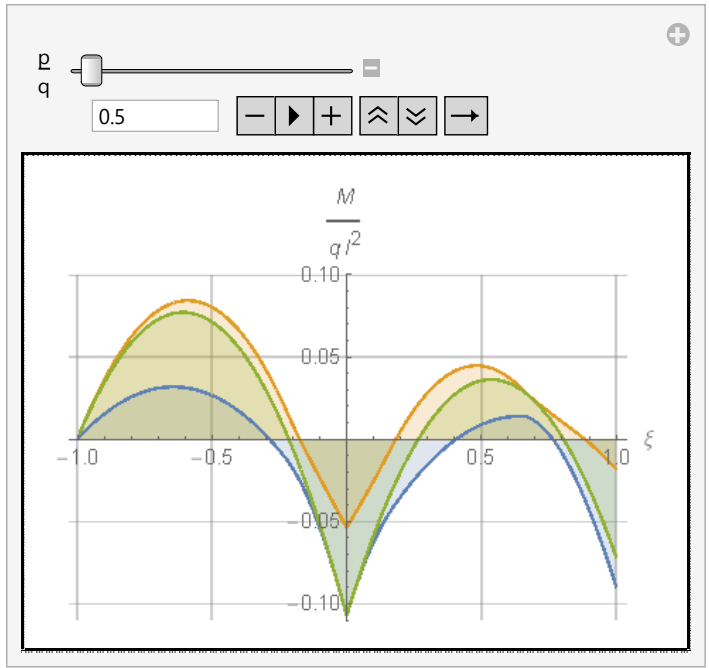

Fig. 31: $M_g(\xi)$, $M_{q+}(\xi)$, $M_{q-}(\xi)$ functions of two span hinged-clamped beam and $\dfrac{p}{q} = \dfrac{1}{2}$

Finally bending moments caused by moving point load on background of the envelopes is presented in Figs. 32-36.
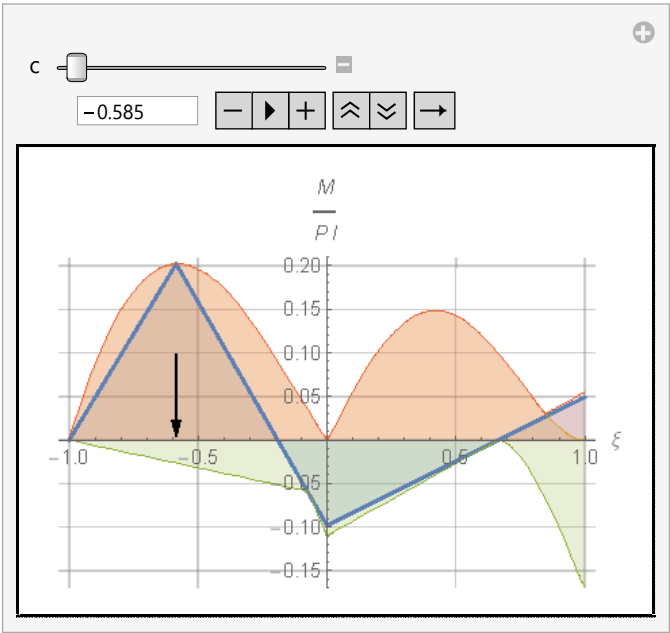

Fig. 32: Bending moment function caused by point load P in position $c = -0.585$ on background of its envelope in two span hinged-clamped beam
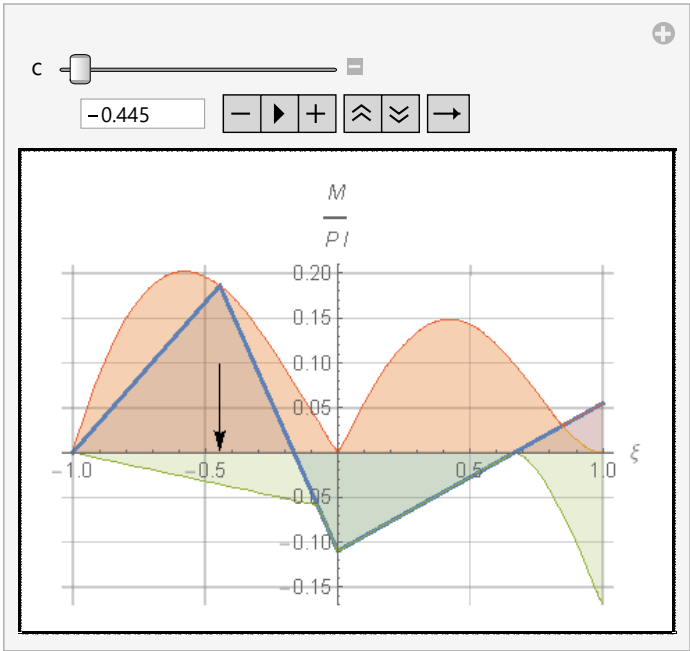

Fig. 33: Bending moment function caused by point load P in position $c = -0.445$ on background of its envelope in two span hinged-clamped beam
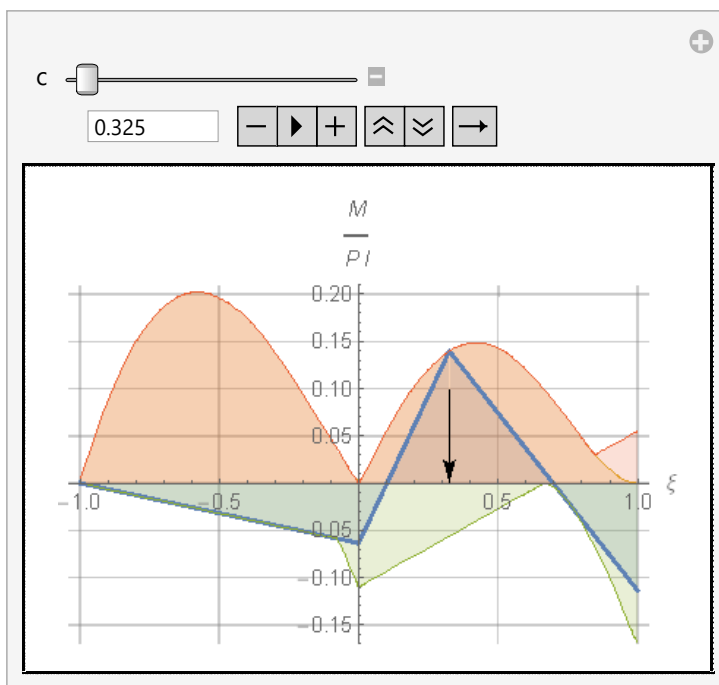
Fig. 34: Bending moment function caused by point load *P* in position $c = 0.325$ on background of its envelope in two span hinged-clamped beam
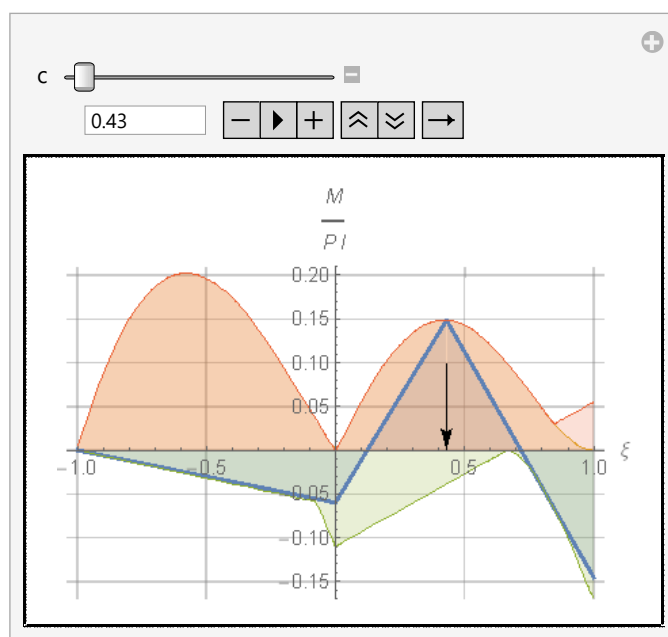

Fig. 35: Bending moment function caused by point load *P* in position $c = 0.43$ on background of its envelope in two span hinged-clamped beam
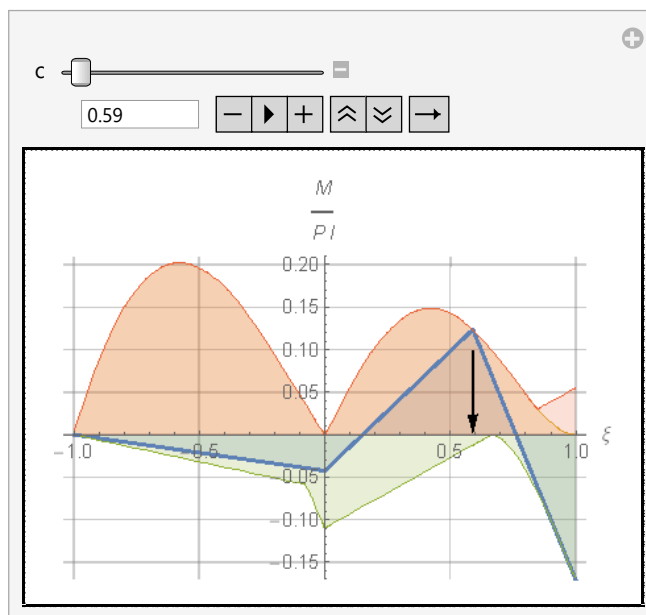
Fig. 36: Bending moment function caused by point load P in position $c = 0.59$ on background of its envelope in two span hinged-clamped beam

6 CONCLUSION

There is presented that thanks to *Mathematica* and implemented into it generalized functions and calculus operators it is possible to introduce an analytical method of evaluation influence lines in statically indeterminate beams. Moreover envelopes of internal forces can be also evaluated analytically.

The proposed approach is a result of computational experiment with fourth order differential equation which right hand side contains higher derivatives of Dirac delta. It has been found that the solution has a physical sense and interesting engineering application.

The presented approach can be applied in engineering practice and education of Structural Engineering students.

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Reviewers:

Doc. Ing. Roland Jančo, PhD., Institute of automation, measurement and applied informatics, Faculty of Mechanical Engineering, Slovak University of Technology in Bratislava, Slovakia.

Ing. Hana Šimonová, Ph.D., Institute of Structural Mechanics, Faculty of Civil Engineering, Brno University of Technology, Czech Republic.