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EXIT CONDITION FOR PROBABILISTIC ASSESSMENT USING MONTE CARLO METHOD

PODMÍNKA UKONČENÍ PRAVDĚPODOBNOSTNÍHO VÝPOČTU PROVÁDĚNÉHO METODOU MONTE CARLO

Abstract

This paper introduces a condition used to exit a probabilistic assessment using the Monte Carlo simulation, and to evaluate it with regard to the relationship between the computed estimate of the probability of failure and the target design probability. The estimation of probability of failure is treated as a random variable, considering its variance that is dependent on the number of performed Monte Carlo simulation steps. After theoretical derivation of the decision condition, it is tested numerically with regard to its accuracy and computational efficiency. The condition is suitable for optimization design using the Monte Carlo method.

Keywords

Monte Carlo simulation, exit condition, probability of failure, design probability, accuracy, convergence, reliability.

Abstrakt

Příspěvek představuje podmínku sloužící k ukončení pravděpodobnostního výpočtu prováděného metodou Monte Carlo a k jeho vyhodnocení z hlediska vztahu mezi vypočteným odhadem pravděpodobností poruchy a návrhovou pravděpodobností. S odhadem pravděpodobnosti poruchy je nakládáno jako s náhodnou veličinou při uvážení jejího teoretického rozptylu v závislosti na provedeném počtu simulačních kroků Monte Carlo. Po teoretickém odvození rozhodovací podmínky následuje její numerické testování z hlediska přesnosti a výpočetní náročnosti. Představená podmínka je použitelná pro optimalizační návrh s využitím metody Monte Carlo.

Klíčová slova

Simulace Monte Carlo, podmínka ukončení, pravděpodobnost poruchy, návrhová pravděpodobnost, přesnost, konvergence, spolehlivost.

1 INTRODUCTION

One of the methods studied for use in probability design and / or assessment of structures is the Monte Carlo simulation method, see e.g. [2]. The basic advantage of this method is its robustness, given i.a. the fact that the accuracy of the method is independent of the dimension of the problem, namely, the number of random input variables. A frequently reported disadvantage is the high

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number of simulation steps required for a sufficiently accurate estimate of the probability of failure P_f, especially for very low P_f values. This issue has been dealt with in a more detailed manner in [1] by one of the authors of this paper.

Following this paper, the authors focused on the possibility to reduce the required number of simulation steps in those applications of Monte Carlo, where there is no need to quantify the probability of failure P_f, but to only determine with sufficient certainty whether this probability is greater or lesser than the specified design probability P_d. Example of the application where this simplified calculation would find its use is the method of probability optimization design of the structures, which has been dealt with by the second author, see e.g. [5].

The paper intends to define the condition under which the Monte Carlo simulation can be prematurely exited and thus its result evaluated, i.e. whether the probability of failure is greater or lesser than the design probability. It also presents the results of numerical testing of the aforementioned condition in the simulation assessments consisting of verification of its accuracy and computational demand.

2 DERIVATION OF CONDITION

Monte Carlo method is in reliability assessment of structures used to calculate the probability of failure estimate P_f^* defined by:

$$P_f^* = \frac{N_f}{N} \tag{1}$$

where:

N_f – number of simulation steps, in which the failure has been detected

N – total number of simulation steps performed

At the same time, it has been shown that with an increasing number of simulation steps performed the hereby calculated estimate of the probability of failure P_f^* converges to the true value of the probability of failure P_f, see e.g. [4].

The probability that a particular step in the Monte Carlo simulation will generate a failure is equal to the searched P_f value, and thus it can be said that the number of failures generated in one simulation step is given by the Bernoulli distribution of A(p) using the parameter $p = P_f$. It also applies that this probability is independent of the results of previous simulation steps. The number of failures N_f generated in N steps is thus a random variable given by the sum N of these distributions, and can be therefore represented by a binomial distribution B(n, p):

$$N_f = \mathrm{B}(n,p) = \mathrm{B}(N, P_f) = \sum_{i=1}^{N} \mathrm{A}_i(P_f) \tag{2}$$

It is known that for a sufficiently high value of the n parameter a binomial distribution can be approximated using the normal distribution N(μ, σ). This fact can be derived from the central limit theorem, under which the sum of a large number of random variables with arbitrary distributions has a normal distribution. As evident from the expression (2), this assumption is met for a sufficiently large number of simulation steps N. For the μ (mean value) and σ (standard deviation) parameters of the obtained normal distribution, the same relations are valid according to [6] as for the approximated binomial distribution B(n, P_f), thus:

$$\mu_{N_f} = N \cdot P_f \tag{3}$$

$$\sigma_{N_f} = \sqrt{N \cdot P_f \cdot (1 - P_f)} \tag{4}$$

As the number of steps when the failure was detected is a random variable with the distribution as described above, the estimate of the probability failure P_f^* is also considered a random variable defined by equation (1). P_f^* has therefore a normal distribution with the parameters:

$$\mu_{P_f^*} = \frac{\mu_{n_f}}{N} = P_f \tag{5}$$

$$\sigma_{P_f^*} = \frac{\sigma_{N_f}}{N} = \sqrt{\frac{P_f \cdot (1 - P_f)}{N}} \tag{6}$$

During the simulation calculation a regularly updated estimate of the probability of failure can be seen as a function of $P_f^*(N)$, thus a random process. The above relation (5) shows that this random process is centered around the probability of failure P_f, and the relation (6) shows that the standard deviation of the process decreases proportionally with $\sqrt{N}$. It is therefore obvious that $P_f^*(N) \rightarrow P_f$ applies for $N \rightarrow \infty$, as stated in the introduction of this chapter.

In the probabilistic assessment the calculated probability of failure P_f (or in case of the Monte Carlo method to use its estimate P_f^*) is compared to the design probability P_d. The evaluated structure is declared reliable if the following condition is met:

$$P_f^* < P_d \tag{7}$$

In the optimization calculation [5] the structure is assessed with varying parameters that affects the probability of failure, while for most combinations of parameters either $P_f >> P_d$, or $P_f << P_d$ applies. With such significant disproportions between the two probabilities there is no need to calculate P_f^* with great accuracy. It is sufficient to be possible to state, with a sufficient degree of certainty, whether or not the condition (7) has been met. This can significantly reduce the number of simulation steps needed throughout the optimization calculation.

The proposed solution leading to this goal, which the authors present herein, assumes that during the simulation assessment the value of the estimated probability of failure P_f^* is being regularly monitored and compared with the design probability P_d. If this estimate is close to the P_d value, the assessment proceeds to the next simulation step. If the value of the estimate moves off by more than a certain defined tolerance ε, the assessment is exited. Thus:

$$\begin{aligned} &P_f^* < P_d - \varepsilon && \Rightarrow \text{ Result: } P_f < P_d. \\ &P_d - \varepsilon < P_f^* < P_d + \varepsilon && \Rightarrow \text{ Continue on with next simulation step.} \\ &P_d + \varepsilon < P_f^* && \Rightarrow \text{ Result: } P_f > P_d. \end{aligned} \tag{8}$$

To calculate the tolerance ε, the following equation is used:

$$\varepsilon = t \cdot \sigma = t \cdot \sqrt{\frac{P_d \cdot (1 - P_d)}{N}} \tag{9}$$

where:

t – standard deviation multiple,

P_d – target design probability,

N – number of simulation steps performed so far.

In the above relation (9) the standard deviation σ calculated by the relation (6) is used, but the design probability P_d applies. It is thus unnecessary to calculate the standard deviation of the $P_f^*(N)$ process, which results in further simplification of the simulation calculation. This simplification is based on the following assumption: if the P_f value is close to the P_d value, the values of tolerance ε are also close to each other if calculated both from the $P_f^*(N)$ and P_d, thus the accuracy of the calculation is not affected. This assumption was verified numerically, the result is shown in Figure 2. On the contrary, if the P_f and P_d values are far from each other, the ε value does not significantly matter, because the simulation calculation will be exited after only a small number of steps, and thus the difference of the absolute number of simulation steps for the ε calculated from $P_f^*(N)$ is not significant.

The presented standard deviation multiple t is dependent on the level of calculation reliability, and therefore, the proper evaluation of the condition (8). It applies, that the greater the t is the greater the certainty is but at the expense of a higher required number of simulation steps, see also Tab. 1. Determination of the t value is based on the width of the confidence interval for normal distribution calculated according to:

$$t = \Phi^{-1}\left(\frac{1+c}{2}\right) \tag{10}$$

where:

Φ^{-1} – inverse distribution function of normal distribution,

c – theoretical level of certainty from interval (0; 1).

During testing of the condition it was shown that a large number of incorrectly assessed simulations was caused by the fact that the estimated probability of failure $P_f^*(N)$ had left $P_d \pm \varepsilon$ boundary after a very small number of steps performed. The authors attribute this behavior to the fact that the central limit theorem does not apply for a small number of steps N, and consequently to the derived relations based on the assumption of the normal distribution of P_f^*. Therefore, an additional parameter was introduced to the exit condition of the simulation, which is the minimum number of simulation steps N_{min} during which the condition is not tested. Details about the N_{min} value and its impact on the average number of steps required as well as on the reliability of the calculation are listed in the following chapter. Example of implementation of such calculation is shown in Figure 1.

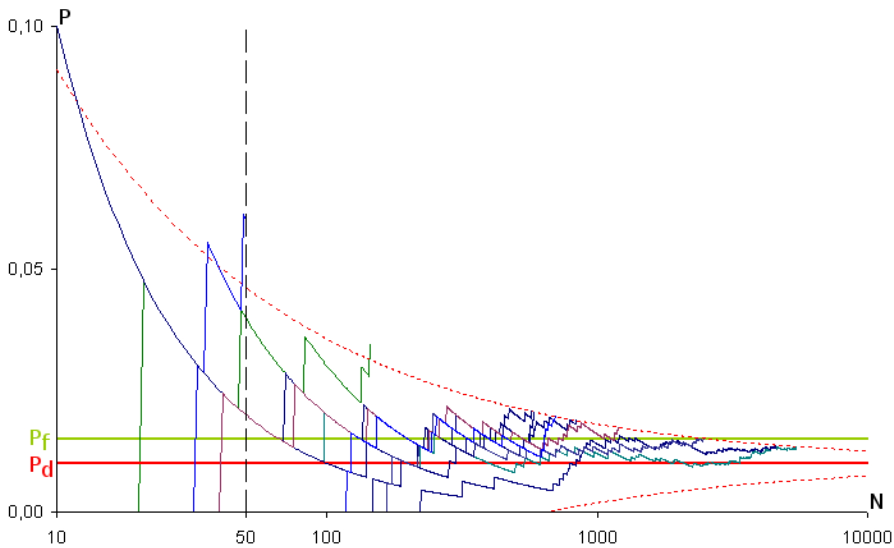

Fig. 1: Example of implementation of the condition for 10 independent simulation runs. The bold red line shows the value of $P_d = 0.010$, the dotted red line shows the $P_d \pm \varepsilon$ boundary, the light green line shows the value of $P_f = 0.015$, the dashed vertical line represents the value of $N_{min} = 50$, before which the condition is not being tested

Furthermore, it is appropriate to establish a maximum number N_{max} of simulation steps, after which the calculation is exited, even if the $P_f^*(N_{max})$ estimate had not left $P_d \pm \varepsilon$ boundaries. This restriction prevents a situation where, in the case of very close values of P_f and P_d, the calculation would lead to a large number of simulation steps. The N_{max} value can be determined in the usual manner as the required number of simulation steps to produce an assessment using the Monte Carlo method, e.g. according to [4]:

$$N_{max} = P_d\left(1 - P_d\right)\left(\frac{t}{\varepsilon}\right)^2 \tag{11}$$

P_d – design probability

t – standard deviation multiple according to the selected level of reliability

ε – acceptable absolute tolerance (half-width of confidence interval).

2 NUMERICAL TESTING OF CONDITION

2.1 Verification of assumption of standard deviation

The first assumption that has been verified is the accuracy of approximation of an estimate of the failure probability P_f^* by the normal distribution with parameters according to the relations (5) and (6). In the Matlab environment, using [3] a total of 100 probabilistic assessments using the Monte Carlo method were simulated. The average μ_{Pf^*} and the standard deviation σ_{Pf^*} were calculated from the values of probability estimates after 10, 100, 1,000 and 10,000 simulation steps. Assuming the normality of estimates of the failure probabilities P_f^*, some 5% and 95% fractiles were consequently derived, which were compared to the boundaries calculated according to the relation (8) for the 90% theoretical level of certainty (c = 0.9). The result of this comparison is shown in Figure 2, from which the good agreement between the two processes is evident.

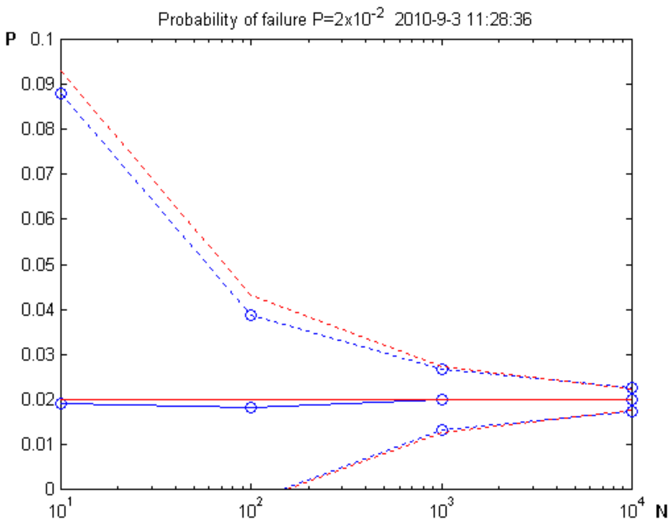

Fig. 2: Verification of the standard deviation assumption; blue values for 100 simulated calculations, the red line represents theoretical values according to relations (8)

2.2 Influence of Theoretical Level of Certainty

To test such an influence, as well as the following effects, the C-language programme was created, which was able to conduct its own simulations and at the same time also allowed for the calculation of necessary characteristics. These characteristics have been exported to text files, whereas the graphs (Fig. 1, 3, 4) were created using Excel software.

A test of influence of the theoretical degree of certainty has always included 10,000 simulation assessments with values of P_f = 0.009; P_d = 0.010 and N_{min} = 50. What was being changed was the multiple value of the standard deviation t to reflect the theoretical levels of certainty (confidence intervals) c listed in Table. 1.

Table 1: Influence of confidence interval width for P_f = 0.009; P_d = 0.010 and N_{min} = 50

c	t	Calculation Reliability Rate (success rate of assessment that $P_f < P_d$)	Average Number of Simulation Steps
0,9	1,645	71,94 %	9,306
0,99	2,576	92,67 %	50,215
0,999	3,291	97,24 %	94,471
0,9999	3,891	99,06 %	139,554

Assessment quality depends on the distance between P_f and P_d – the greater the difference between the probabilities is, the more successful the decision-making procedure is.

2.3 Influence of Minimum Number of Simulation Steps

Another tested parameter was the minimum number of simulation steps N_{min} during which the calculation exit condition is not verified. As in the previous test, 10,000 simulation calculations with the values of P_f = 0.009, P_d = 0.010, c = 0.99 (t = 2.576) were always used. What was being changed was the value of N_{min} steps listed in Table 2 and Figure 3. Interestingly enough, the recommended number of steps using the Monte Carlo simulation for the above listed parameters according to (10) amounts to 100,000 to 200,000 steps. It is apparent that even despite the high value of N_{min} = 10,000, the fact that $P_f < P_d$ is evaluated with almost one hundred percent success rate after only about 60,000 steps on average.

Table 2: Influence of the minimum number of simulation steps for P_f = 0.009; P_d = 0.010 and t = 2576

N_{min}	Calculation Reliability Rate (success rate of assessment that $P_f < P_d$)	Average Number of Simulation Steps
10	83,70%	44,989
20	90,04%	48,029
50	92,67%	50,215
100	94,19%	51,169
200	95,83%	52,147
500	97,57%	53,513
1000	98,77%	54,473
2000	99,15%	54,556
5000	99,75%	55,711
10000	99,95%	58,452

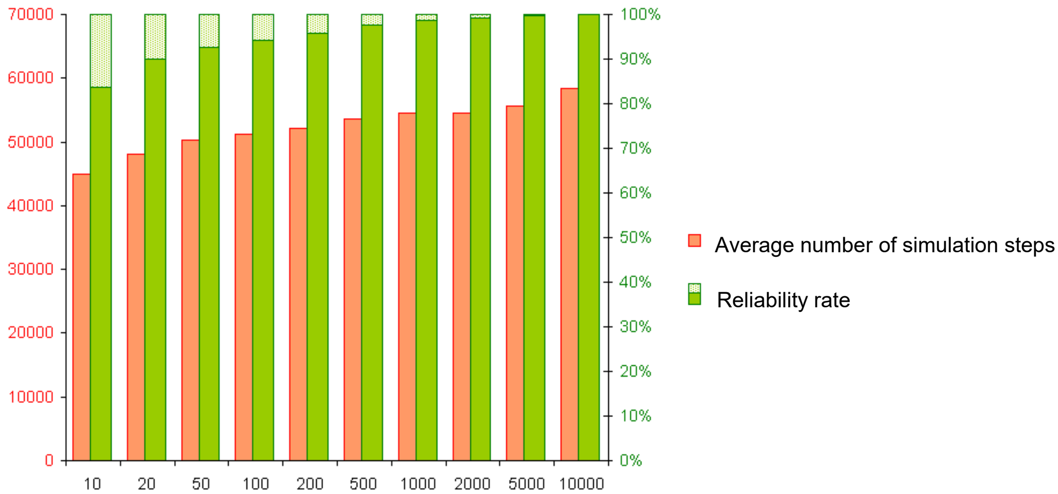

Figure 3: Influence of the minimum number of simulation steps N_{min} on the average number of simulation steps (left vertical axis) and on calculation of the reliability rate (expressed as the proportion of green filled columns)

2.4 Distribution of Number of Simulation Steps

In the previous Tables 1 and 2 the average number of simulation steps is shown after which the condition was found to be met and the simulation calculation was exited. An interesting characteristic which may help improve the image of the operation of the condition is the shape of the distribution describing the number of simulation steps. This distribution, for the values of $P_f = 0.015$, $P_d = 0.010$, $t = 2.576$ and $N_{min} = 50$, is shown using the histogram in Figure 4. An average number of simulation steps performed until the exit of the simulation run was 2,053 in this case. The pronounced distribution asymmetry is quite apparent, with approximately 62% of the values are below the average.

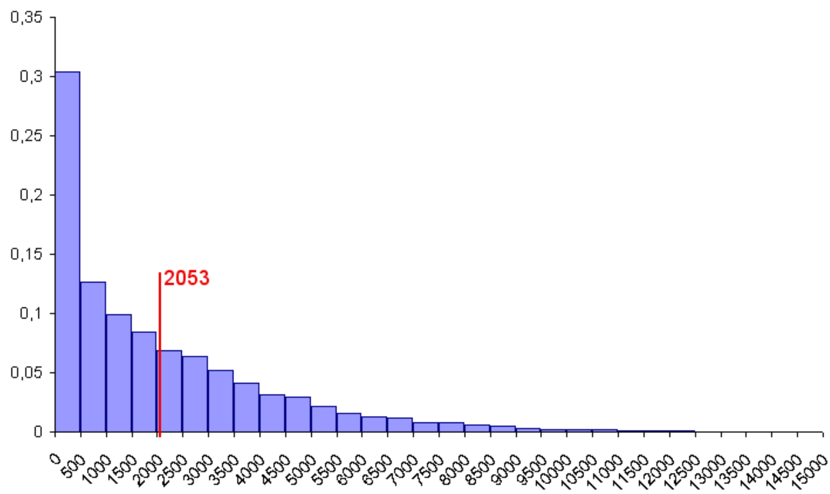

Figure 4: Histogram showing the number of steps, after which the simulation was exited, with the average value in red

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3 CONCLUSION

The paper introduces the condition intended to exit the Monte Carlo simulation. The aim of the authors is to reduce the required number of simulation steps in cases when the probability of failure P_f significantly differs from the design probability P_d.

While deriving the condition, the simplification is introduced and justified which consists in the fact that the distance of boundaries ε is not calculated from the standard deviation of the estimate of the probability of failure P_f^* but the design probability P_d is placed into the relationship derived from the condition.

The condition introduced herein is applicable to accelerate the optimization calculation using the Monte Carlo simulation. Its numerical testing revealed that the quality of evaluation depends on the selected rate of reliability of the estimate, expressed by the t parameter, and on the minimum number N_{min} of simulation steps, where the condition is not tested because the assumptions on which the condition was derived do not apply with sufficient accuracy.

The selected approach will be further tested in the optimization calculations. The knowledge gained will be used to specify the parameters of the presented condition, or to specify the definition of the condition itself for a small number of simulation steps.

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