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Jakub SOBEK[1], Václav VESELÝ[2], Lucie ŠESTÁKOVÁ[3]

ACCURACY OF APPROXIMATION OF STRESS AND DISPLACEMENT FIELDS IN CRACKED BODY FOR ESTIMATION OF FAILURE ZONE EXTENT

PŘESNOST APROXIMACE POLÍ NAPĚTÍ A POSUNŮ V TĚLESE S TRHLINOU PRO ODHAD ROZSAHU ZÓNY PORUŠENÍ

Abstract

The paper deals with an analysis of the stress and displacement fields in a cracked body. The intention of the authors is to determine the sufficient number of terms of the Williams power series for an accurate approximation of the near-crack-tip fields which can be subsequently used *e.g.* for estimation of the extent of the fracture process zone in quasi-brittle materials. Values of coefficients of these terms are determined via regression from results of numerical simulations; the coefficients are expressed as functions of the relative crack length. The analysis is conducted on a 2D numerical model of the wedge-splitting test on a modified standard cube-shaped specimen used commonly for testing of cementitious composites; ANSYS FE computational system is employed.

Keywords

Cracked body, near-crack-tip fields, Williams power series, higher-order terms, FEM, fracture process zone, quasi-brittle fracture.

Abstrakt

Příspěvek se zabývá analýzou polí napětí a deformací v tělese s trhlinou. Záměrem autorů je určení dostačujícího počtu členů Williamsova mocninného rozvoje pro přesnou aproximaci těchto polí v okolí vrcholu trhliny využitelnou dále např. pro odhad rozsahu lomové procesní zóny v kvazikřehkých materiálech. Hodnoty koeficientů těchto členů jsou určovány regresí z výsledků numerických simulací; jsou vyjádřeny jako funkce relativní délky trhliny. Řešeným případem je lomový test štípání klínem upraveného standardního krychlového tělesa pro zkoušení cementových kompozitů; numerická studie je provedena v MKP výpočetním systému ANSYS.

Klíčová slova

Těleso s trhlinou, pole v okolí trhliny, Williamsova mocninná řada, členy vyšších řádů, MKP, lomová procesní zóna, kvazikřehký lom.

1 INTRODUCTION, GOALS

Fracture behaviour of the quasi-brittle building materials (*e.g.* concrete, ceramics) is studied via experiments on test specimens with stress concentrators (especially notches). In these tests (destructive ones in most cases) the specimens are loaded in different tensile/compressive/bending modes in order to determine the fracture-mechanical properties and parameters of models which are

[1] Ing. Jakub Sobek, Institute of Structural Mechanics, Faculty of Civil Engineering, Brno University of Technology, Veveří 331/95, 602 00 Brno, tel.: (+420) 541 147 116, e-mail: sobek.j@fce.vutbr.cz.

[2] Ing. Václav Veselý, Ph.D., ditto, tel.: (+420) 541 147 362, e-mail: vesely.v1@fce.vutbr.cz.

[3] Ing. Lucie Šestáková, Ph.D., ditto, tel.: (+420) 541 147 362, e-mail: sestakova.l@fce.vutbr.cz.

utilized for numerical simulations of the fracture phenomena. It should be noted that these tests are very time consuming and also very costly. Obviously, the motivation of the research is to reduce the number of required real tests for estimation of fracture properties of the investigated material by using numerical simulations. In other words, relatively less expensive numerical modelling can be employed for investigation of different aspects of failure phenomena occurring in tests on real specimens.

Description of fracture of quasi-brittle materials is very problematic (reviewed *e.g.* in [1−4]). It has been shown that for the correct interpretation of the tensile failure phenomena in quasi-brittle materials the existence of the so-called fracture process zone (FPZ) in front of the crack tip must be considered. The FPZ changes its shape and size during the crack propagation [5−9]. It is also necessary to take into account the boundary conditions in the analysis of the numerical model of the test specimen which considerably influence the FPZ extent. The investigation of the FPZ existence and its characteristics is still an actual research topic, both in the experimental [10−15] and computational [15−18] field.

In the recent studies by the authors [19−23], the suitability of the regression technique, referred to as the Over-Deterministic Method (ODM [24]), utilized to obtain the values of the coefficients of the higher members of the Williams power series [25] for bodies with cracks during the Wedge-splitting Test (WST) was tested. Principle of the ODM is based on the solution of an over-deterministic systems of algebraic equations (application of the least-squares method) expressed from the formulae for the displacement field in a body with crack via Williams power series. Coordinates of selected set of nodes of the used finite element mesh as well as their displacements (calculated *e.g.* using common commercial finite-element software) serve as inputs to the system of equations. Obtained values of the coefficients of the terms of Williams series can be then expressed as functions of the crack length. After specification of these values by the level of the applied stress, the dimensionless expressions of the coefficients of the series' terms are obtained, which are referred to as the shape functions. These functions can then be re-used for analytical reconstruction of the stress and displacement fields (by placing its values into the Williams power series expressions).

In this paper, the authors deal with the accuracy of the above-mentioned shape functions, especially for the higher-order terms of the Williams power expansion (which characterize the influence of applied boundary conditions). Several variants of the selection of nodes near the crack tip (their number and the distance from the crack tip) were considered.

2 STRESS AND DISPLACEMENT FIELDS IN THE BODY WITH CRACK

2.1 Williams power expansion

The stress and displacement fields in a homogeneous isotropic body with a crack can be expressed as an infinite power series − Williams expansion [25]. The stress tensor $\{\sigma\}$ and deformation vector $\{u\}$ can be written for the isotropic elastic body with a crack as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} A_n \cdot \begin{Bmatrix} \left[2+(-1)^n+\frac{n}{2}\right]\cos\left(\frac{n}{2}-1\right)\theta - \left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-3\right)\theta \\ \left[2-(-1)^n-\frac{n}{2}\right]\cos\left(\frac{n}{2}-1\right)\theta + \left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-3\right)\theta \\ -\left[(-1)^n+\frac{n}{2}\right]\sin\left(\frac{n}{2}-1\right)\theta + \left(\frac{n}{2}-1\right)\sin\left(\frac{n}{2}-3\right)\theta \end{Bmatrix}, \tag{1}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \sum_{n=1}^{\infty} \frac{r^{n/2}}{2\mu} A_n \cdot \begin{Bmatrix} \left(\kappa + \frac{n}{2} + (-1)^n\right)\cos\frac{n}{2}\theta - \frac{n}{2}\cos\left(\frac{n}{2}-2\right)\theta \\ \left(\kappa - \frac{n}{2} - (-1)^n\right)\sin\frac{n}{2}\theta + \frac{n}{2}\sin\left(\frac{n}{2}-2\right)\theta \end{Bmatrix}, \tag{2}$$

where:

r, θ – polar coordinates (the origin of the coordinate system is situated in the crack tip, the x axis is oriented in the direction of the crack propagation) [m],

μ – shear modulus ($\mu = E/(2(1 + \nu))$ [Pa],

κ – Kolosov's constant ($\kappa = (3 - \nu)/(1 + \nu)$ for plane stress and $\kappa = (3 - 4\nu)$ for plane strain) [-],

E, ν – Young's modulus and Poisson's ratio, [Pa] and [-], respectively,

A_n – coefficients of the terms of the series, constants for a particular value of the crack length [Pa/m$^{n/2-1}$], and

n – index of term of the power expansion [-].

Values of coefficients A_n can be expressed as functions of the relative crack length and are normed by the applied load – thus the dimensionless shape function g_n are defined [26–27]. Following formulas correspond to the individual coefficients of terms of the Williams series:

$$g_n(\alpha) = \frac{A_n(\alpha)}{\sigma} W^{\frac{n-2}{2}} \quad \text{for } n = 1, 3, 4 \ldots, N \quad \text{and} \quad g_2 = t(\alpha) = \frac{4A_2(\alpha)}{\sigma}, \tag{3}$$

where:

α – is the relative crack length ($\alpha = a/W_{\text{ef}}$) [-],

σ – is the nominal stress in the central plane of the specimen caused by the applied load ($\sigma = P_{\text{sp}}/BW$) [Pa],

W – is width of the test specimen [m], and

B – is its breadth [m].

The individual dimensions of the WST specimen are indicated in the sketch of the test configuration in Fig. 1a.

2.2 Over-Deterministic Method

The computational technique which is used for solving the system of equations resulting from eq. (2) is referred to as the Over-Deterministic Method (ODM, [24]). Mathematically, the ODM is based on the formulation of linear least-squares, and its aim is the solution of the system of $2k$ equations, where k represents the number of selected nodes around the crack tip for first N selected terms of the power series. From knowledge of the components of the displacement vector u a v in k selected nodes of the finite-element mesh (typically from a finite elemental solution using common available computational software) and from knowledge of the coordinates of these nodes, we can evaluate equation (2) for N terms of the series, so that $N \leq 2k$. The solution of the resulting over-deterministic system of equations (4) by the least-squares formulation (schematically written in equation (5) and (6)) leads to the vector of coefficients of terms of the series A_n (respectively the corresponding shape functions g_n):

$$\{U\} = [F] \cdot \{A\} \tag{4}$$

$$[F]^T \cdot \{U\} = [F]^T \cdot [F] \cdot \{A\} \tag{5}$$

$$\{A\} = \left([F]^T \cdot [F]\right)^{-1} \cdot [F]^T \cdot \{U\} \tag{6}$$

where:

$\{U\}$ – represents vector of displacements u and v in chosen set of nodes of the FE mesh,

$[F]$ – is matrix of geometrical functions, dependent on the polar coordinates of FE nodes (see eq. (2)),

$\{A\}$ – represents vector of the desirable coefficients of the Williams expansion.

3 NUMERICAL ANALYSIS

3.1 Computational model

For the analysis of the stress and displacement fields, a numerical model of a typical test specimen used for wedge splitting test (WST, [28]) was considered. It is a cube with the edge length equal to 100 mm provided with a groove and a notch. The test geometry with description of the specimen dimensions is shown in Fig. 1a. The configuration with one support (located in the center of the bottom side) and two components of the loading force, acting on the steel loading plate (the load transmitted via the wedge can be decomposed into the horizontal and the vertical force, see Fig. 1b). was considered

The numerical model was created in ANSYS FE computational system [29]. The propagating crack (of zero width) was simulated by keeping the nodes related to the crack face free of any constraint; whereas all degrees of freedom of the nodes on the resting ligament were removed (a symmetric half of model was used). The FEM model was created from the eight-node quadrilateral isoparametric finite elements (PLANE 82); the crack-tip singularity was taken into account via using triangular crack elements with the mid-side node moved to 1/4 of the length of the triangle side (see details in Fig. 2 right). The elastic isotropic continuum was used for both of the material model for test specimen (concrete, modulus of elasticity E = 35 GPa, Poisson's ratio ν = 0.2) and for steel plates (E = 210 GPa, ν = 0.3).

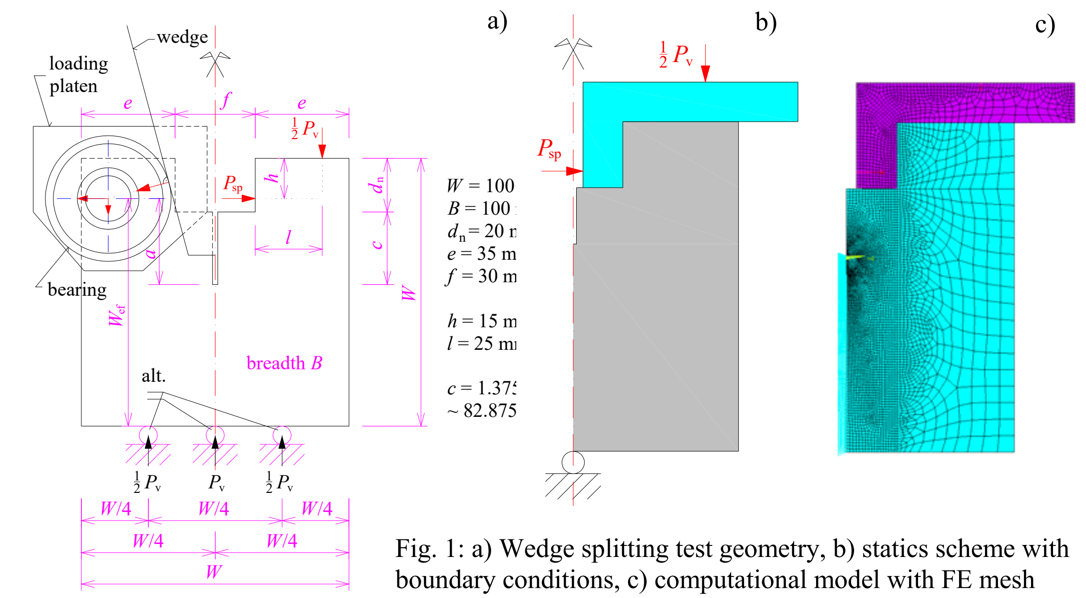

Fig. 1: a) Wedge splitting test geometry, b) statics scheme with boundary conditions, c) computational model with FE mesh

3.2 Variants of conducted study

As was mentioned above, for the calculation of the shape functions g_n related to the specific relative crack length α (by ODM), it is necessary to select a set of nodes of the FE mesh. There were considered several variants of the FE nodes selection in the present numerical study, they varied in count of the nodes and their distance from the crack tip: Three variants of the node selection set size k

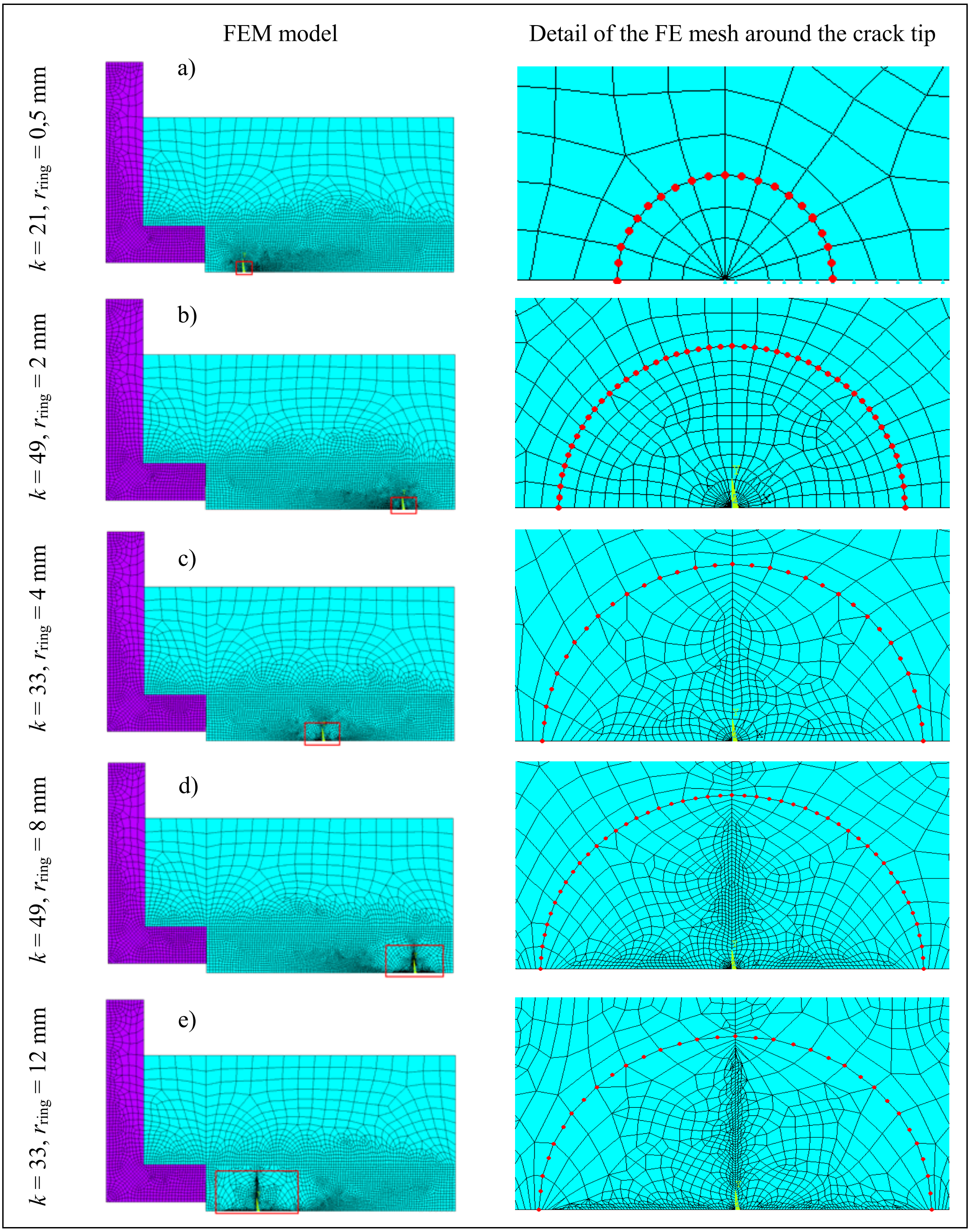

Fig. 2: The used finite element mesh for the entire model (left) and the detail of the mesh around the crack tip (right) for selected variants of the analysis: a) 21 nodes with radius of 0.5 mm, b) 49 nodes with radius of 2 mm, c) 33 nodes with radius of 4 mm, d) 49 nodes with radius of 8 mm, c) 33 nodes with radius of 12 mm

were taken into account (21, 33 and 49 nodes, which correspond to 10, 16 and 24 equal parts of the area around the crack tip, see Fig. 2). The distance from the crack tip (or radius of the ring for the selection of FE nodes r_{ring}) was considered in the values of 0.5, 2, 4, 8 and 12 mm. Selected variants of the used numerical models for different crack length are shown in Fig. 2, together with details of the FE mesh and monitored nodes around the crack tip. Note that the numerical model was made parametrically (using macros) for the purpose of the automatic run of the FEM analyses for different values of the crack length during which the computed results were recorded into an output data file. For some values of the model defining parameters (especially for the relative distance of the nodal ring from the crack tip greater than 4%), an unsuitable shape of the FE mesh has been observed. However, the used computational tool (sw. ANSYS) reported no inappropriate mesh for these variants. Note that the absolute expression of the values of the ring radius r_{ring} used in this paper have to be considered in relation to the length of the cube edge W. Equivalently, it would be also possible to use the relative lengths of the radii (normalized by the W value).

The ODM technique was implemented in MathCAD, further evaluation was performed in MS Excel.

4 DISCUSSION OF RESULTS, CONCLUSION

Evaluation of all 15 considered variants of the model is summarized in Tab. 1 which shows the highest index n of the constructed shape function g_n that can be used for a sufficiently accurate approximation of the stress and displacement fields.

The process of determination of the value of this index is illustrated via comparison of Fig. 3 and Fig. 4, where the curves (shape functions g_n depending on the relative crack length α) are compared for all considered variants of the selected node numbers and their distances from the crack tip – a smooth curves represent the sufficiently accurate result of the selection variants, whereas the fluctuation of the values of the function indicates significant relative error. Fig. 3 illustrates the course of the g_4 shape function for all considered configurations. There can be seen that all of them form a "smooth" curve – *i.e.* results of any of the fifteen considered variants can be used for the (re)construction of the stress and displacement fields in the specimen (using the first four shape functions/terms of Williams expansion). Contrary, significant fluctuations of the curves of shape functions for variants which have the distance of the selected nodes from the crack tip $r_{\text{ring}} = 0.5$ mm can be seen in Fig. 4. Fig. 5 and Fig. 6 display only those functions that have smooth course (the other unsuitable variants are removed). In this way, the highest index of the sufficiently accurate shape function was selected for Tab. 1.

The determination of the sufficient accuracy of approximation is therefore based only on visual monitoring of the shape function course. However, this technique proved suitable in the present analyzes of this research task. And contrary, the "unbiased" criteria (based on the assessment of the accuracy of some type of normalized expression) fails due to the large gradient of the shape functions at the edges of their definition interval (normalization by the absolute value is not feasible), or due to very low values of the functions in some points of the definition interval (for values of functions close to zero − relative normalization is not feasible).

Tab. 1: The highest index n of the sufficiently accurate shape function for various FE nodes selections

$k \setminus r_{\text{ring}}$	0.5 mm	2 mm	4 mm	8 mm	12 mm
21	4	5	7	8	9
33	4	6	8	9	10
49	4	7	9	10	-

From the comparison of the graphs in these figures it is clear that the increase in number of the selected nodes leads better accuracy of the desired shape functions of the coefficients of the higher terms of the series. The larger is the distance from the crack tip r_{ring} the more this effect is pronounced. In other words, it is necessary to take into account results for variants with a larger number of nodes k when higher maximum index n of the series is desired (at the requirement of maintaining a sufficient accuracy). This fact holds, in particular, for variants with greater distances r_{ring} of the FE node selection from the crack tip.

A possible disadvantage of the nodes selection from a greater distance from the crack tip consists in a limited range of the relative crack length for which the shape function can be evaluated. Given the way of the FEM mesh creation around the crack tip (a semicircular area, at boundaries of whose the nodes for ODM are selected), the values of the shape functions for very short and very long cracks are not calculated in the FE analysis (see such a mesh in Fig. 2d and 2e). This disadvantage, according to the authors, can be resolved by creating a procedure for selecting the nodes from a more flexibly defined area. A technique of the FE nodes selection, defined in the

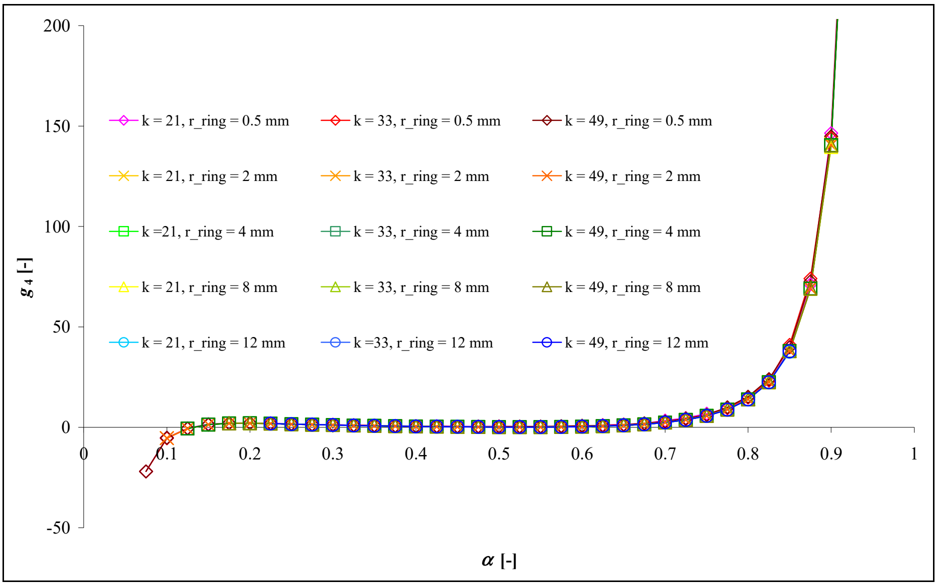

Fig. 3: Course of the g_4 shape function for all considered variants of the FE nodes selection

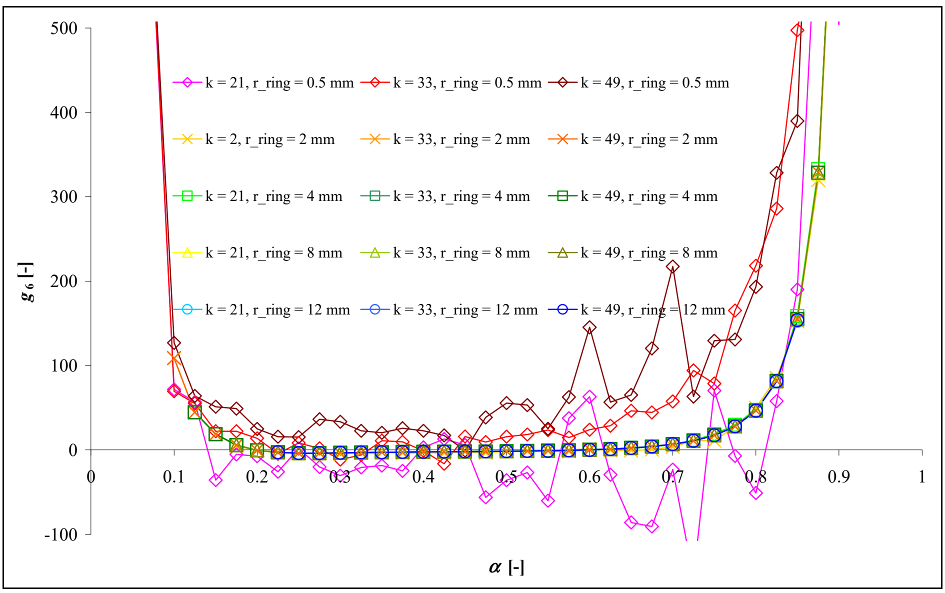

Fig. 4: Course of the g_6 shape function for all the considered variants of the FE node selection

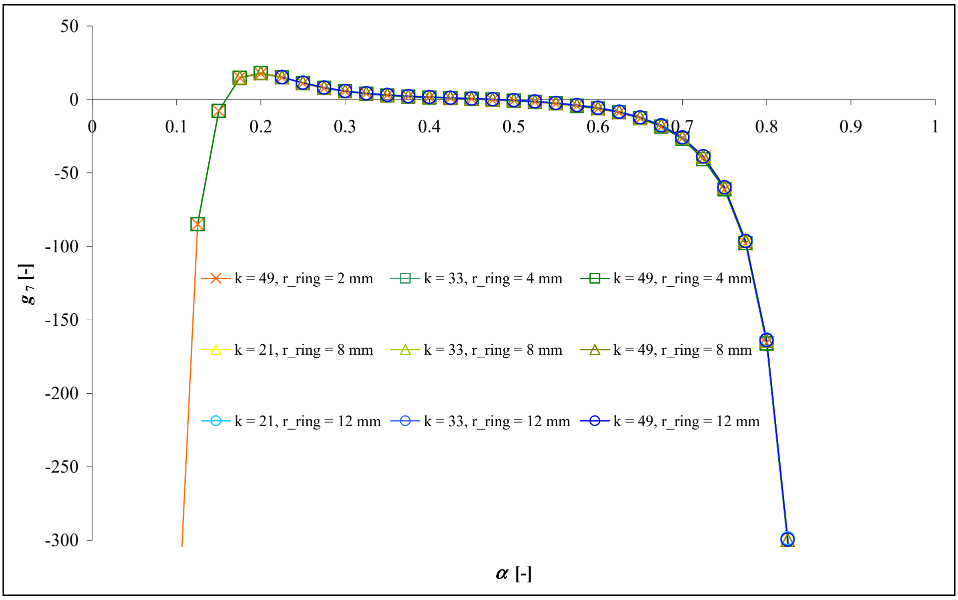

Fig. 5: Course of the g_7 shape functions for the accurate enough variants of the FE nodes selection

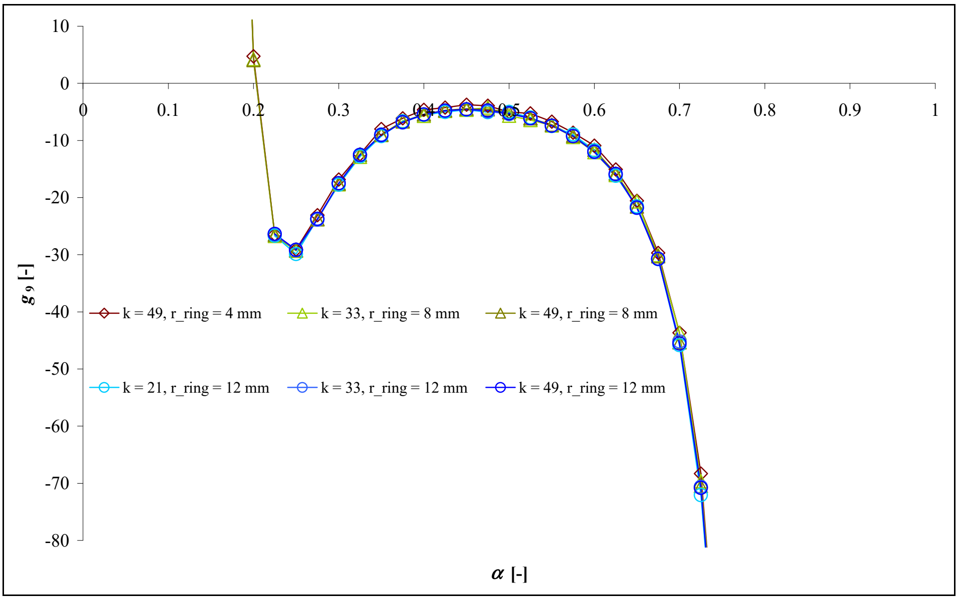

Fig. 6: Course of the g_9 shape function for the accurate enough variants of the FE nodes selection

computational environment of the FEM program ANSYS, will be based on the length of the crack/ligament (*i.e.* the distance from the crack tip to the front or back surface of the body) or other free boundaries of the body. A possibility of improving the obtained results by using multiple layers, from which the nodes are selected, will be investigated. This procedure is currently being developed.

The relevance of the obtained results will be further investigated using alternative ways of the FE mesh creation, especially in the vicinity of the crack tip. Note that the use of ODM for determination of the coefficients of the higher order terms of the Williams power series for approximation of the stress and displacement fields in the body with a crack must be further explored. It shall be done particularly with regard to the accuracy of the coefficients of terms with high indices (*e.g.* $n > 10$) and the necessity/appropriateness of their use in analytical reconstruction of these fields to estimate the extent of the zone of material failure. Analysis of the reconstructed stress fields with their comparison with the numerical solution is currently being processed and prepared for publication.

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Reviewers:

Doc. Ing. Luboš Náhlík, Ph.D., Institute of Physics of Materials of the ASCR, v. v. i.

Ing. Daniel Vavřík, Ph.D., Institute of Theoretical and Applied Mechanics of the ASCR, v. v. i.