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FRAME ON ELASTIC FOUNDATION

RÁM NA PRUŽNOM PODLOŽÍ

Abstract

This article deals with calculation of the beam and frame rested on elastic foundation using matrix displacement method. An exact stiffness matrix of a beam element on elastic foundation is formulated. The beam is rested on elastic Winkler foundation. At the end off paper, same results of frame on elastic foundation are presented.

Keywords

Winkler foundation, elastic foundation, stiffness matrix, beam, frame, displacement method.

Abstrakt

Tento článok sa zaoberá riešením nosníka a rámu uloženého na pružnom podloží s využitím deformačnej metódy. Bola sformulovaná matica tuhosti nosníka na pružnom podloží. Nosník je uložený na pružnom Winklerovom podloží. V závere článku je prezentovaný príklad rámu na pružnom podloží.

Klúčové slová

Winklerovo podložie, pružné podložie, matica tuhosti, nosník, rám, deformačná metóda.

1 INTRODUCTION

When analyse any top building it is necessary to consider also a contact with the foundation. An analysis of such structures is presented by quite a lot of authors, in particular, we mention Jendželovský [6], Mistríková, Frydrýšek [4], Jančo [5], Kollár [3], Eisenberger, Yankelevsky [1] and others. Foundation is unlimited, however, regarding a certain area in contact with the structure of a finite part. Nowadays, one-parametric Winkler´s model of elastic foundation belongs to the most common and used in engineering praxis, Pasternak´s and Vlasov´s models (but also Filonenko-Boroditsch´s, Hetényi´s and others) belong to two-parametric models, introducing the influence of surround foundation through shear forces. On the other hand joined models based on the theory of elastic half-space belong to the other wide class of models, the best known is the Boussinesque´s relationship between deformation and stress in elastic half-space. Today, mainly in last decades, the matrix formulation of models mentioned above as well as Finite Element Method, thanks to computer utilization, are the most used in an analysis of interaction of structure-foundation.

In this paper an exact stiffness matrix of a beam element on elastic foundation is formulated. A more detailed derivation of stiffness matrix of a beam rested on Winkler´s foundation is mentioned in Eisenberger-Yankelevsky [1] and grids on Winkler´s foundation in Kollár-Djubeková [2].

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2 BENDING CURVE OF A BEAM ON WINKLER´S FOUNDATION

In this case a relationship between the resistance of the soil p and the deflection of a beam *w* in terms of Winkler model is presented as follows

$$p(x) = k\ w(x). \tag{1}$$

The coefficient *k* can be determined according to the theorem

$$k = K\ b \tag{2}$$

where

K – is the compressibility modulus of the soil (kNm^{-3})

w – is the deflection of the beam.

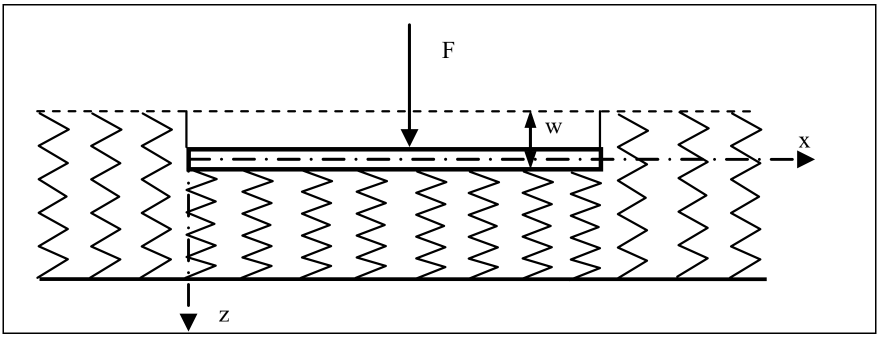

Fig. 1 Deformation of the foundation in the Winkler´s model

Thus, a reaction *p*, which is directly proportional to vertical displacement of foundation w, appears in each point under a loaded beam. This model deflects only where load is applied.

Foundation modulus of elasticity depends mainly on mechanical properties of the soil. Its value for the different types of subsoil varies from $0{,}5.e^4$ (soft clay) to $200.e^4$ kNm^{-3} (compressed gravel). Relationship (1) is valid only for the hydrostatic subsoil with small beam deflections.

The differential equation for the deflection curve of a beam on an elastic foundation is

$$EI_y \frac{d^4 w(x)}{dx^4} = q(x) - p(x)\,. \tag{3}$$

A solution of this differential equation is easier after the introduction of the ξ coordinate.

$$\xi = \frac{x}{\omega} \quad \rightarrow \quad d\xi = \frac{1}{\omega} dx\,, \tag{4}$$

then the differential equation (3) takes form

$$\frac{d^4 w(\xi)}{d\xi^4} + \frac{k\,\omega^4}{EI_y}\ w(x) = \frac{q(\xi)}{EI_y}\,. \tag{5}$$

Coefficient of ω is chosen as follows

$$\frac{k\,\omega^4}{EI_y} = 4 \qquad \rightarrow \qquad \sqrt[4]{\frac{4\,E\,I_y}{K\,b}} = \sqrt[4]{\frac{4\,E\,I_y}{k}} = \omega \quad [m]. \tag{6}$$

The general solution of equation (3) can be rewritten in terms

$$w(\xi) = \sum_{i=1}^{4} A_i\ \Omega_i(\xi) + \overline{w}(\xi)\,, \tag{7}$$

where

$$
\begin{aligned}
\Omega_1(\xi) &= \cos\xi \cosh\xi \\
\Omega_2(\xi) &= \frac{1}{2}\left(\sin\xi \cosh\xi + \cos\xi \sinh\xi\right) \\
\Omega_3(\xi) &= \frac{1}{2}\sin\xi \sinh\xi \\
\Omega_4(\xi) &= \frac{1}{4}\left(\sin\xi \cosh\xi - \cos\xi \sinh\xi\right)
\end{aligned} \tag{8}
$$

$$
A_1(\xi) = w_0, \quad A_2(\xi) = \omega\varphi_0, \quad A_3(\xi) = -\frac{\omega^2}{EI_y} M_0, \quad A_4(\xi) = -\frac{\omega^3}{EI_y} V_0, \tag{9}
$$

Functions $\Omega_i(\xi)$ (8) satisfy the Cauchy´s conditions of initial parameters, that mean, these functions and their derivative fulfil a unit matrix for $\xi = 0$. Coefficients A_i (9) are the initial parameters of deflection in point $\xi = 0$.

Deformation and internal force in any point are expressed as

$$
\varphi(\xi) = \frac{1}{\omega}\frac{dw(\xi)}{d\xi} = -\frac{4}{\omega} A_1\,\Omega_4(\xi) + \frac{1}{\omega} A_2\,\Omega_1(\xi) + \frac{1}{\omega} A_3\,\Omega_2(\xi) + \frac{1}{\omega} A_4\,\Omega_3(\xi) + \frac{1}{\omega}\frac{\overline{w}(\xi)}{d\xi} \tag{10}
$$

$$
\begin{aligned}
M_y(\xi) = -\frac{EI_y}{\omega^2}\frac{dw^2(\xi)}{d\xi^2} = \frac{4EI_y}{\omega^2} A_1\,\Omega_3(\xi) + \frac{4EI_y}{\omega^2} A_2\,\Omega_4(\xi) - \frac{4EI_y}{\omega^2} A_3\,\Omega_1(\xi) - \\
\frac{4EI_y}{\omega^2} A_4\,\Omega_4(\xi) - \frac{EI_y}{\omega^2}\frac{d^2\overline{w}(\xi)}{d\xi^2}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
V_z(\xi) = -\frac{EI_y}{\omega^3}\frac{dw^3(\xi)}{d\xi^3} = \frac{4EI_y}{\omega^3} A_1\,\Omega_2(\xi) + \frac{4EI_y}{\omega^3} A_2\,\Omega_3(\xi) + \frac{4EI_y}{\omega^3} A_3\,\Omega_4(\xi) - \\
\frac{4EI_y}{\omega^3} A_4\,\Omega_1(\xi) - \frac{EI_y}{\omega^3}\frac{d^3\overline{w}(\xi)}{d\xi^3}
\end{aligned} \tag{12}
$$

3 STIFFNESS MATRIX OF BEAMS ON ELASTIC FOUNDATION

Let us consider a double-side fixed beam rested on elastic foundation. Introducing a mathematic model a clockwise coordinate system *x, y, z* is used. No load, except the subsoil resistance, appears on the section *a-b*. Due to the load of the neighbouring sections of the beam, the nodes *a*, *b* as well as the beam are deflected. At nodes *a*, *b* are some nodal forces F_{ab}^z, F_{ba}^z and nodal bending moments M_{ab}^y, M_{ba}^y corresponding with nodal deflections and rotations w_a, φ_{ya} and w_b, φ_{yb}, as seen in Fig.2.

The deflection curve of a beam on elastic foundation is given in equations (7) and (10). The terms of the stiffness matrix, are defined as the generalized force reactions at the ends of the beam, due to unit translations and rotations in given in formulas (7) a (10). Since the beam does not have any load, particular integrals $\overline{w}(\xi), \overline{\varphi}_y(\xi)$ will be equal zero.

From

$$
\xi = \frac{x}{\omega} = 0 \quad \rightarrow \quad w(0) = w_a = -1 \quad a \rightarrow \quad \varphi(0) = \varphi_a = 0
$$

$$
\xi = \frac{L}{\omega} = \lambda \quad \rightarrow \quad w(\lambda) = w_b = 0 \quad a \rightarrow \quad \varphi(\lambda) = \varphi_b = 0 \tag{13}
$$

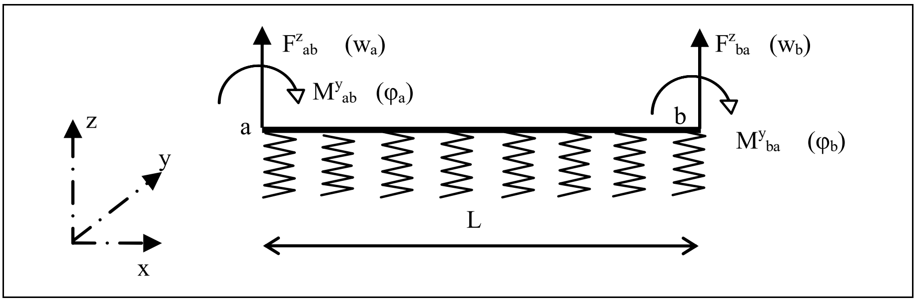

Fig. 2 Member rested on elastic foundation

one may calculate the integration coefficients A_i and after their introducing into the equations (11) and (12) get the first column of the stiffness matrix for the member fixed on the both sides.

Similarly, we determine the second column of the stiffness matrix for the boundary conditions (14)

$$
\begin{aligned}
\xi &= \frac{x}{\omega} = 0 \quad \rightarrow \quad w(0) = w_a = 0 \quad a \rightarrow \quad \varphi(0) = \varphi_a = 1 \\
\xi &= \frac{L}{\omega} = \lambda \quad \rightarrow \quad w(\lambda) = w_b = 0 \quad a \rightarrow \quad \varphi(\lambda) = \varphi_b = 0
\end{aligned}
\tag{14}
$$

The third column of the matrix is on the basis of boundary conditions (15)

$$
\begin{aligned}
\xi &= \frac{x}{\omega} = 0 \quad \rightarrow \quad w(0) = w_a = 0 \quad a \rightarrow \quad \varphi(0) = \varphi_a = 0 \\
\xi &= \frac{L}{\omega} = \lambda \quad \rightarrow \quad w(\lambda) = w_b = -1 \quad a \rightarrow \quad \varphi(\lambda) = \varphi_b = 0
\end{aligned}
, \tag{15}
$$

and the fourth column due to (16).

$$
\begin{aligned}
\xi &= \frac{x}{\omega} = 0 \quad \rightarrow \quad w(0) = w_a = 0 \quad a \rightarrow \quad \varphi(0) = \varphi_a = 0 \\
\xi &= \frac{L}{\omega} = \lambda \quad \rightarrow \quad w(\lambda) = w_b = 0 \quad a \rightarrow \quad \varphi(\lambda) = \varphi_b = 1
\end{aligned}
\tag{16}
$$

The following derived matrix equation for the beam on elastic foundation has the form

$$
\begin{Bmatrix} F^0_{za} \\ M^0_{ya} \\ F^0_{zbzb} \\ M^0_{yb} \end{Bmatrix} = \begin{bmatrix} \eta_1 & \eta_3 & -\eta_6 & \eta_4 \\ \eta_3 & \eta_5 & -\eta_4 & \eta_7 \\ -\eta_6 & -\eta_4 & \eta_1 & -\eta_3 \\ \eta_4 & \eta_7 & -\eta_3 & \eta_5 \end{bmatrix} \begin{Bmatrix} w_a \\ \varphi_{ya} \\ w_b \\ \varphi_{yb} \end{Bmatrix} + \begin{Bmatrix} \overline{F}^0_{za} \\ \overline{M}^0_{ya} \\ \overline{F}^0_{zb} \\ \overline{M}^0_{yb} \end{Bmatrix} , \tag{17}
$$

where individual members are

$$
\eta_1 = \frac{4EI_y}{L^3}\lambda^3 \frac{\sinh\lambda\cosh\lambda + \cos\lambda\sin\lambda}{\sinh^2\lambda - \sin^2\lambda}
$$

$$
\eta_3 = \frac{2EI_y}{L^2}\lambda^2 \frac{\sinh^2\lambda + \sin^2\lambda}{\sinh^2\lambda - \sin^2\lambda}
$$

$$
\eta_4 = \frac{4EI_y}{L^2}\lambda^2 \frac{\sinh\lambda\ \sin\lambda}{\sinh^2\lambda - \sin^2\lambda}
$$

$$\eta_5 = \frac{2EI_y}{L}\lambda\frac{\sinh\lambda\cosh\lambda+\cos\lambda\sin\lambda}{\sinh^2\lambda-\sin^2\lambda}$$

$$\eta_6 = \frac{2EI_y}{L^3}\lambda^3\frac{\sin\lambda\cosh\lambda+\cos\lambda\sinh\lambda}{\sinh^2\lambda-\sin^2\lambda} \tag{18}$$

$$\eta_7 = \frac{2EI_y}{L}\lambda\frac{\sin\lambda\cosh\lambda+\cos\lambda\sinh\lambda}{\sinh^2\lambda-\sin^2\lambda} \ .$$

If there is an external load, then the expressions $\overline{F}_{za}, \overline{M}_{ya}, \overline{F}_{zb}, \overline{M}_{yb}$ are the nodal forces and nodal bending moments of a beam fixed on the both sides rested on elastic foundation from the effects of particular external load. We can also identify them by the Kollár, Djubeková [3].

4 MATRIX DISPLACEMENT METHOD

Consider a simple frame with the beam on elastic foundation in Fig. 2. For columns and beams the classical member stiffness matrix in local coordinate system of both sides clamped member is introduced as follows

$$\mathbf{k}^0 = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_y}{L^3} & \frac{-6EI_y}{L^2} & 0 & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} \\ 0 & \frac{-6EI_y}{L^2} & \frac{4EI_y}{L} & 0 & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & 0 & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\ 0 & -\frac{6EI_y}{L^2} & \frac{2EI_y}{L} & 0 & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} \end{bmatrix} \tag{19}$$

For the transformation from the local to the global coordinate system, the transformation matrix A for rotation is used

$$\mathbf{A} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & -\cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 & -\sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -\cos\alpha & -\sin\alpha & 0 & \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & -\cos\alpha & 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \ . \tag{20}$$

A member stiffness matrix in the global coordinate system is determined as follows

$$\mathbf{k} = \mathbf{A}^T\mathbf{k}^0\ \mathbf{A} \ . \tag{21}$$

And the transformation of force vector from LCS to GCS is by regulation

$$\overline{\boldsymbol{F}} = \mathbf{A}^T\overline{\boldsymbol{F}}^0 \ . \tag{22}$$

The displacements on the whole structure are determined from the system of linear algebraic equations, after the introduction of boundary conditions

$$\mathbf{K}\ \boldsymbol{\delta} = \boldsymbol{RS} \ , \tag{23}$$

where $\boldsymbol{RS}$ we have identified as follows

$$\boldsymbol{RS} = \boldsymbol{R} - \sum \overline{\boldsymbol{F}} \,. \tag{24}$$

After the determination of the displacement vector from the equation (21) we can calculate the nodal force vectors for any member in GCS as follows

$$\boldsymbol{F} = \overline{\boldsymbol{F}} + \mathbf{k}\,\boldsymbol{\delta} \,. \tag{25}$$

The transformation of nodal forces vectors from GCS to LCS is by regulation

$$\boldsymbol{F}^{0} = \mathbf{A}\,\boldsymbol{F} \,. \tag{26}$$

Simultaneously, the transformation of the ending forces to the internal forces (27).

$$\begin{Bmatrix} N_a \\ V_a \\ M_{ya} \\ N_b \\ V_b \\ M_{yb} \end{Bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} F^{0}_{xa} \\ F^{0}_{za} \\ M^{0}_{ya} \\ F^{0}_{xb} \\ F^{0}_{zb} \\ M^{0}_{yb} \end{Bmatrix} \tag{27}$$

5 FRAME ON ELASTIC FOUNDATION

For numerical analysis we consider a simple frame rested on elastic foundation (see Fig.3). The dimensions of columns are 0,3 x 0,3 m, of the beam is 0,3 x 0,4 m, made of concrete (B12.5), modulus of elasticity is 21 GPa.

Foundation beam is considered by dimensions 1 x 1,2 m, made of the same concrete, rested on Winkler´s foundation with the coefficient of the subsoil of 75 MN/m^2 (medium compressed gravel and sand).

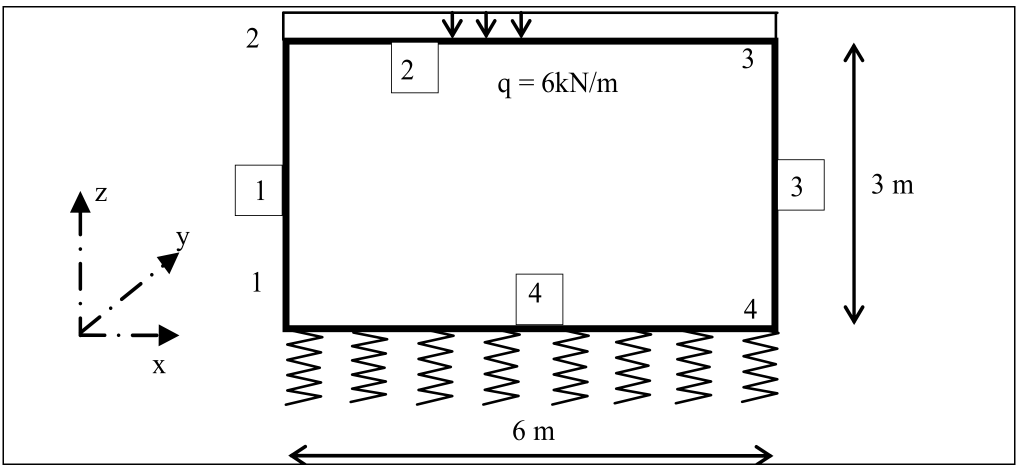

Fig.3 Frame made of concrete rested on Winkler´s foundation

5.1 Calculation by generalized displacement method

The frame is solved by general displacement method in matrix formulation described in section 4. Stiffness matrices for members *1*, *2* and *3* shall be considered due to (19).

For the beam *4* rested on elastic foundation we consider the stiffness matrix due to (17), supplemented with horizontal nodal forces into matrix

$$\begin{Bmatrix} F_{xa}^{0} \\ F_{za}^{0} \\ M_{ya}^{0} \\ F_{xb}^{0} \\ F_{zb}^{0} \\ M_{yb}^{0} \end{Bmatrix} = \begin{bmatrix} \eta_2 & 0 & 0 & -\eta_2 & 0 & 0 \\ 0 & \eta_1 & \eta_3 & 0 & -\eta_6 & \eta_4 \\ 0 & \eta_3 & \eta_5 & 0 & -\eta_4 & \eta_7 \\ -\eta_2 & 0 & 0 & \eta_2 & 0 & 0 \\ 0 & -\eta_6 & -\eta_4 & 0 & \eta_1 & -\eta_3 \\ 0 & \eta_4 & \eta_7 & 0 & -\eta_3 & \eta_5 \end{bmatrix} \begin{Bmatrix} u_a^0 \\ w_{aa}^0 \\ \varphi_{ya}^0 \\ u_b^0 \\ w_b^0 \\ \varphi_{yb}^0 \end{Bmatrix}, \tag{28}$$

where $\eta_2 = \dfrac{EA}{L}$.

The resulting internal forces on members are shown in Tab. 1.

Tab. 1: Internal forces on the frame

	member 1	member 2	member 3	member 4
Na(kN)	-36,00	-8,22	-36,00	0,00
Va(kN)	-8,22	36,00	8,22	-36,00
Ma(kNm)	8,20	-16,45	-16,45	-8,20
Nb(kN)	-36,00	-8,22	-36,00	0,00
Vb(kN)	-8,20	-36,00	8,22	36,00
M2(kNm)	-16,45	-16,45	8,20	-8,20

In nodes *1* and *4*, the vertical displacement is $w = -1{,}93.e^{-4}$m and in nodes *2* and *3* $w = -1{,}14.e^{-3}$ m.

5.2 Solution using the program Nexis

For numerical solution we use the program Nexis based on using the FEM. Boundary conditions are as follows: the displacement at node *1* and *4* in the *x*-direction is equal zero. Properties and the characteristics of cross-sectional area are entered in state 5.

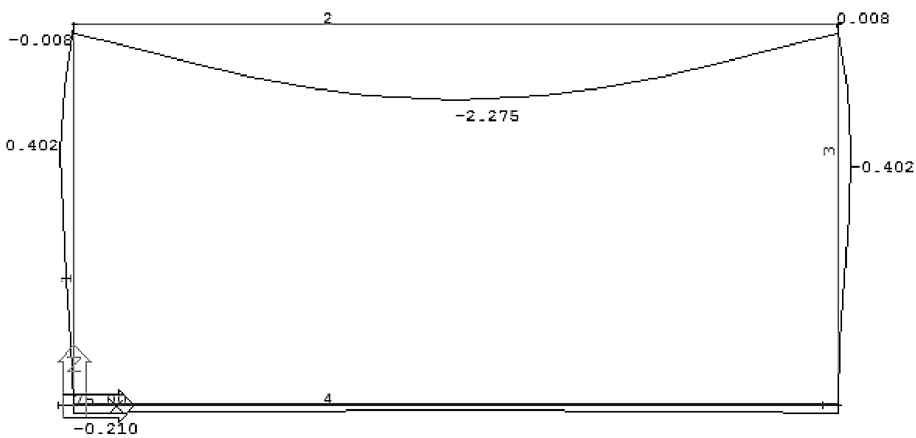

Fig. 4 Diagram of deflections of frame on elastic foundation in milimeters

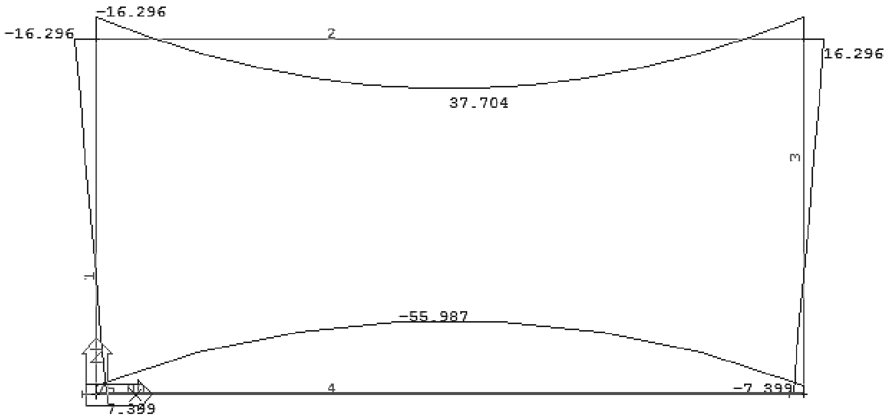

Fig. 5 Diagram of bending moments of the frame in *kNm*

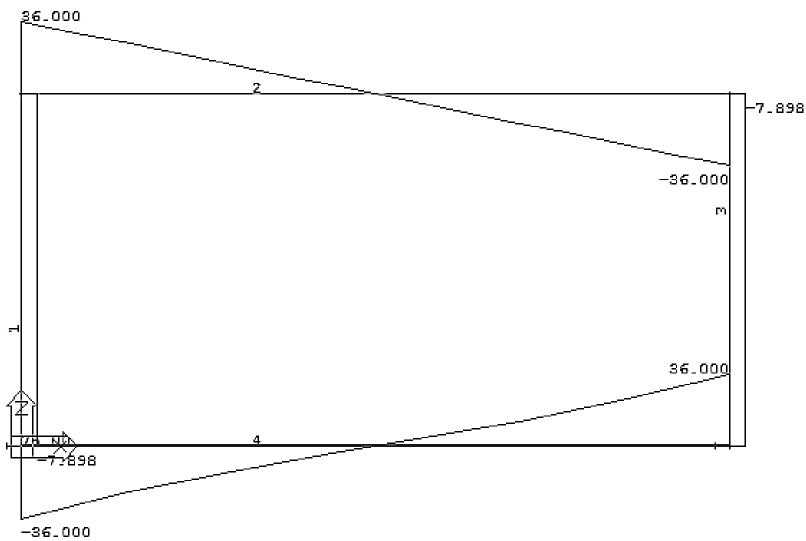

Fig. 6 Diagram of shear forces of the frame in kN

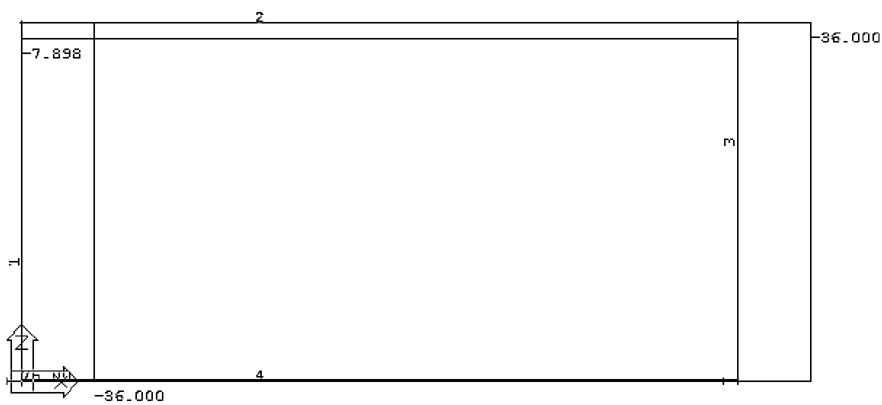

Fig. 7 Diagram of normal forces for the frame rested on Winkler foundation in *kN*

6 CONCLUSION

An analysis of frames and beams rested on elastic foundation is quite a frequent task. However, the stiffness matrix of a beam on elastic foundation can be quite easy incorporated into the calculation of the frames by using general matrix methods. Today, computers are usually already dealt with through such structures using FEM, as shown in Jančo [5] or Kormaníková, Kotrasová [7], [8].

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