

Maciej MAJOR¹, Izabela MAJOR²ACCELERATION WAVE PROPAGATING IN HYPERELASTIC MOONEY-RIVLIN
AND ZAHORSKI MATERIALS**Abstract**

The paper presents the problems of propagation of the acceleration wave in a cylinder made of hyperelastic incompressible Mooney – Rivlin and Zahorski materials. The study analysed the velocity of acceleration wave propagation and the shape of the front of propagating surface of discontinuities. The analytical results obtained in the study were presented graphically.

Keywords:

Acceleration wave, hyperelastic materials, rubber.

1 INTRODUCTION

Let us consider a circular cylinder made of an incompressible elastic material with the initial radii R and R_1 , with $R < R_1$. We use a cylindrical coordinate system of $\{X^\alpha\} = \{R, \Theta, Z\}$ with the axis X^3 coinciding with the cylinder axis. For the present configuration, we also use a circular coordinate system of $\{x^i\} = \{r, \vartheta, z\}$. The initial deformation is given by the following formulas [3]:

$$r = \sqrt{R^2 + c}; \quad \vartheta = \Theta; \quad z = Z \quad (1)$$

where c is a parameter that describes the deformation.

Equations (1) describe a cylinder's inflation where its length remains invariable whereas the incompressibility condition is met with an identity.

For the deformation (1), we know the deformation gradient and the left B^{ik} and right $C_{\alpha\beta}$ deformation tensors:

$$[x^i_\alpha] = \begin{bmatrix} \frac{R}{r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad [B^{ik}] = \begin{bmatrix} \frac{R^2}{r^2} & 0 & 0 \\ 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad [C_{\alpha\beta}] = \begin{bmatrix} \frac{R^2}{r^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2)$$

The invariants of the deformation tensor are:

$$I_1 = \frac{R^2}{r^2} + \frac{r^2}{R^2} + 1; \quad I_2 = I_1; \quad I_3 = \frac{R^2}{r^2} \cdot \frac{r^2}{R^2} \cdot 1 = 1 \quad (3)$$

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The equation of motion for the initial deformation (1) of an incompressible material is given by (see [3])

$$A_i^{\alpha k \beta} x^k \Big|_{\alpha\beta} + p \Big|_{\alpha} X^{\alpha}_{,i} = 0 \quad (4)$$

In this case, the covariant derivative of the scalar reduces to a partial derivative. Neither the scalar nor absolute derivative depends on the spatial basis. After transformation, the equation is given by:

$$A_i^{\alpha k \beta} x^k \Big|_{\alpha\beta} + p_{,i} = 0 \quad (5)$$

The equation (5) is a basic equation which should be met by the deformation (1). The first-order tensor for the elastic material function has the coordinates (see [4]):

$$\begin{aligned} A_i^{\alpha k \beta} &= 2\sigma_1 g_{ik} g^{\alpha\beta} + \\ &+ 2\sigma_2 \left[2x_i^{\alpha} x_k^{\beta} - g_{ik} C^{\alpha\beta} - x_i^{\alpha} x_k^{\beta} + (I_1 g_{ik} - B_{ik}) g^{\alpha\beta} \right] + 4\sigma_{11} x_i^{\alpha} x_k^{\beta} \end{aligned} \quad (6)$$

For further calculations, we adopt the Zahorski's material characterized by the elastic constants of C_1, C_2, C_3 , [1]

$$W(I_1, I_2) = \sigma p_R = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(I_1^2 - 9) \quad (7)$$

For that material, the values of the derivatives $\sigma_1, \sigma_2, \sigma_{11}$ are:

$$\sigma_1 = \frac{1}{\rho_R} (C_1 + 2C_3 I_1); \quad \sigma_2 = \frac{C_2}{\rho_R}; \quad \sigma_{11} = \frac{2C_3}{\rho_R} \quad (8)$$

An ordinary differential equation for the function p is:

$$\frac{dp}{dR} = 2(C_1 + C_2) \left[-\frac{R(R^2 + 2c)}{(R^2 + c)^2} + \frac{1}{R} \right] - 4C_3 \left[\frac{2R}{R^2 + c} + \frac{3R^3 c}{(R^2 + c)^3} - \frac{2}{R} - \frac{c}{R^3} \right] \quad (9)$$

Solving the above equation yields an equation of pressure p (see [2]):

$$\begin{aligned} p &= p_o + c(C_1 + C_2) \left[\frac{1}{R^2 + c} - \frac{1}{R_o^2 + c} \right] + \\ &+ C_3 c \left[3 \left(\frac{c + 2R^2}{(R^2 + c)^2} - \frac{c + 2R_o^2}{(R_o^2 + c)^2} \right) - 2 \left(\frac{1}{R^2} - \frac{1}{R_o^2} \right) \right] + \\ &+ \ln \left(\frac{R^2 (R_o^2 + c)}{R_o^2 (R^2 + c)} \right)^{(C_1 + C_2 + 4C_3)} \end{aligned} \quad (10)$$

Assuming that $C_3 = 0$ gives an equation of p for the Mooney – Rivlin material (see [4]).

2 THE CONDITION OF PROPAGATION OF THE DISTURBANCE IN THE FORM OF AN ACCELERATION WAVE

We limit our considerations to an axially symmetric surface of discontinuities Σ . The equation for this surface is given by (see [4]):

$$t = \psi(r, z) \quad (11)$$

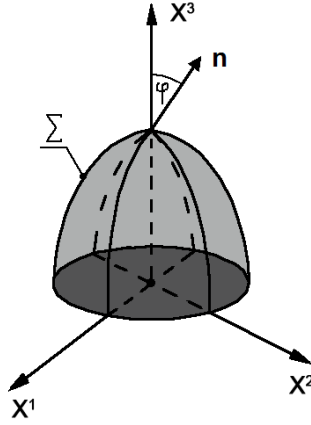


Fig. 1: Axially symmetric surface of discontinuities Σ in the $\{x^i\}$ coordinate system and a normal vector \mathbf{n} with a slope angle φ

According to [4], the acoustic tensor q_{ik} depends on the direction of propagation \mathbf{n} :

$$q_{ik} = A_i^{\alpha} k^{\beta} x^r_{\alpha} x^s_{\beta} n_r n_s \quad (12)$$

The reduced acoustic tensor q_{ik}^* is given by the relation (see [4]):

$$q_{ik}^* = q_{ik} - q_{rk} n_i n^r \quad (13)$$

The acoustic tensor q_{ik} and reduced acoustic tensor q_{ik}^* are:

$$[q_{ik}] = \begin{bmatrix} q_{11} & 0 & q_{13} \\ 0 & q_{22} & 0 \\ q_{31} & 0 & q_{33} \end{bmatrix}; \quad [q_{ik}^*] = \begin{bmatrix} q_{11}^* & 0 & q_{13}^* \\ 0 & q_{22}^* & 0 \\ q_{31}^* & 0 & q_{33}^* \end{bmatrix} \quad (14)$$

Non-zero coordinates of both the acoustic tensor and reduced acoustic tensor are:

$$\begin{aligned} q_{11} &= 2 \cos^2 \varphi (\Psi_6 + \Psi_1) + \frac{2R^2 \sin^2 \varphi}{r^2} (\Psi_2 + \Psi_1) \\ q_{22} &= 2R^2 \sin^2 \varphi (\Psi_9 + \Psi_1) + 2r^2 \cos^2 \varphi (\Psi_8 + \Psi_1) \\ q_{33} &= \frac{2R^2 \sin^2 \varphi}{r^2} (\Psi_6 + \Psi_1) + 2 \cos^2 \varphi (\Psi_7 + \Psi_4) \\ q_{13} &= q_{31} = \frac{R^2}{r^2} \Psi_5 \sin 2\varphi \\ q_{11}^* &= \frac{2 \cos^2 \varphi}{r^2} \left[R^2 (\Psi_2 + \Psi_3) \sin^2 \varphi - R^2 \Psi_5 \sin^2 \varphi + r^2 (\Psi_6 + \Psi_1) \cos^2 \varphi \right] \\ q_{22}^* &= 2 \left[R^2 (\Psi_9 + \Psi_1) \sin^2 \varphi + r^2 (\Psi_8 + \Psi_1) \cos^2 \varphi \right] \\ q_{33}^* &= \frac{2 \sin^2 \varphi}{r^2} \left[R^2 (\Psi_6 + \Psi_1) \sin^2 \varphi + r^2 (\Psi_7 + \Psi_4) \cos^2 \varphi - R^2 \Psi_5 \cos^2 \varphi \right] \end{aligned} \quad (15)$$

$$q_{13}^* = -\frac{\sin 2\varphi}{r^2} \left[R^2(\Psi_6 + \Psi_1) \sin^2 \varphi + r^2(\Psi_7 + \Psi_4) \cos^2 \varphi - R^2 \Psi_5 \cos^2 \varphi \right]$$

$$q_{31}^* = -\frac{\sin 2\varphi}{r^2} \left[R^2(\Psi_2 + \Psi_3) \sin^2 \varphi - R^2 \Psi_5 \sin^2 \varphi + r^2(\Psi_6 + \Psi_1) \cos^2 \varphi \right]$$

where:

$$\begin{aligned} \Psi_1 &= 2C_3 \left(\frac{R^2}{r^2} + \frac{r^2}{R^2} + 1 \right); & \Psi_2 &= C_1 + C_2 \left(\frac{r^2}{R^2} + 1 \right); & \Psi_3 &= 2C_3 \left(\frac{3R^2}{r^2} + \frac{r^2}{R^2} + 1 \right) \\ \Psi_4 &= C_3 \left(\frac{R^2}{r^2} + \frac{r^2}{R^2} + 5 \right); & \Psi_5 &= C_2 + 4C_3; & \Psi_6 &= C_1 + C_2 \frac{r^2}{R^2} \\ \Psi_7 &= C_1 + C_2 \left(\frac{R^2}{r^2} + \frac{r^2}{R^2} \right); & \Psi_8 &= C_1 + C_2 \frac{R^2}{r^2}; & \Psi_9 &= C_1 + C_2 \end{aligned} \quad (16)$$

The propagation velocities are given by:

$$u_1 = \sqrt{\frac{2}{\rho r^2} \left\{ \cos^2 \varphi \left[R^2(\Psi_2 + \Psi_3) \sin^2 \varphi - R^2 \Psi_5 \sin^2 \varphi + r^2(\Psi_6 + \Psi_1) \cos^2 \varphi \right] + \sin^2 \varphi \left[R^2(\Psi_6 + \Psi_1) \sin^2 \varphi - R^2 \Psi_5 \cos^2 \varphi + r^2(\Psi_7 + \Psi_4) \cos^2 \varphi \right] \right\}}$$

$$u_2 = 0 \quad (17)$$

$$u_3 = \sqrt{\frac{2 \left[R^2(\Psi_9 + \Psi_1) \sin^2 \varphi + r^2(\Psi_8 + \Psi_1) \cos^2 \varphi \right]}{\rho r^2}}$$

3 THE FRONT OF THE ACCELERATION WAVE IN THE ZAHORSKI MATERIAL

In our case, the reduced acoustic tensor (14)₂ is symmetrical. Substitution of the coordinate values q_{ik}^* to the propagation condition for the velocity of the acceleration wave in an incompressible material yields (see [4]):

$$\left(q_{ik}^* - \rho u^2 g_{ik} \right) a^k = 0 \quad (18)$$

Note that there are two possible amplitudes (see [4])

$$a^{(1)k} = \left[0, \frac{1}{r}, 0 \right]; \quad a^{(2)k} = [\cos \varphi, 0, -\sin \varphi] \quad (19)$$

determined with the accuracy to the scalar multiplier. These amplitudes are mutually orthogonal and orthogonal with respect to the vector normal to the surface of discontinuities Σ . Both amplitudes represent transverse waves (which are the only waves possible in an incompressible material). By substituting vector coordinates \mathbf{n} and (16) to (17)₃, we obtain the differential equation for the function $\psi(r, z)$ that represents the wave front. With $C_3 = 0$ we obtain an expression for a Mooney – Rivlin material that has been widely used in the literature [4]

$$\frac{2}{\rho r^2} \left[R^2 \left[C_1 + C_2 + 2C_3 \left(\frac{R^2}{r^2} + \frac{r^2}{R^2} + 1 \right) \right] \psi_r^2 + \left[C_1 + \frac{R^2}{r^2} C_2 + 2C_3 \left(\frac{R^2}{r^2} + \frac{r^2}{R^2} + 1 \right) \right] \psi_z^2 \right] = 1 \quad (20)$$

If there is no initial deformation ($c = 0$), then $R = r$ and the equation (17)₃ reduces to

$$c_o^2 = \frac{2}{\rho}(C_1 + C_2 + 6C_3) \quad (21)$$

This demonstrates that the propagation velocity depends neither on the point nor on the direction of propagation. The initial deformation gives priority to certain directions and favours certain points in the space. If $c < 0$ (radial compression $r < R$), then, for each φ , we obtain $u_3 > c_o$ while u_3 is a monotonically decreasing function of r . If $c > 0$ (radial extension $r > R$) then, for each φ , we obtain $u_3 < c_o$ while u_3 is a monotonically increasing function of r . For a given r , the extreme value of c (maximum) corresponds to $\varphi = \frac{\pi}{2}$. We introduce the following denotations:

$$M_1 = \frac{2C_3}{C_1 + C_2 + 6C_3}; \quad M_2 = \frac{C_2}{C_1 + C_2 + 6C_3} \quad (22)$$

By substituting (22) to (20) and given that $R^2 = r^2 - c$ and (21), we obtain a non-linear partial differential equation of the first order:

$$c_o^2 \left[\left(1 - \frac{c}{r^2} \right) \left[1 + M_1 \left(\frac{c^2}{r^2(r^2 - c)} \right) \right] \psi_r^2 + \right. \\ \left. + \left(1 - M_2 \frac{c}{r^2} + M_1 \frac{c^2}{r^2(r^2 - c)} \right) \psi_z^2 \right] = 1 \quad (23)$$

We solve this equation using the small parameter method by limitation to the first two approximations, and assuming that the constant c represents the small parameter describing the basic deformation (1):

$$\psi(r, z) = \overset{o}{\psi}(r, z) + c \overset{1}{\psi}(r, z) \quad (24)$$

By substituting (24) to the equation and equating the coefficients of the respective powers of the parameter c to zero, we obtain:

$$c_o^2 \left(\overset{o}{\psi}_r^2 + \overset{o}{\psi}_z^2 \right) = 1; \quad 2 \left(\overset{o}{\psi}_r \overset{1}{\psi}_r + \overset{o}{\psi}_z \overset{1}{\psi}_z \right) = \frac{1}{r^2} \left(\overset{o}{\psi}_r^2 + M_2 \overset{o}{\psi}_z^2 \right) \quad (25)$$

We are interested in the surface area Σ which, at the instant $t = 0$ is a plane with $z = 0$, that propagates towards the cylinder axis. As shown by the numerical analysis, this surface is gradually deformed as it moves towards the axis (see [4]).

According to the above assumption $\left(\overset{o}{\psi}_r = 0 \right)$ and the equation (9), $\overset{o}{\psi}$ for zero approximations is:

$$\overset{o}{\psi} = \frac{1}{c_o} z \quad (26)$$

Substitution of (26) to the equation (25)₂ yields a linear equation for the approximation $\overset{1}{\psi}$. The solution to this equation is

$$\overset{1}{\psi} = \frac{1}{c_o} \left(\frac{M_2}{2r^2} \right) z + g(r) \quad (27)$$

where $g(r)$ is arbitrary. For the surface of discontinuities Σ represents the plane $z=0$, hence $g(r)=0$. With the accuracy to the linear expressions with respect to c the surface of discontinuities $\psi(r, z)$ is given by the function

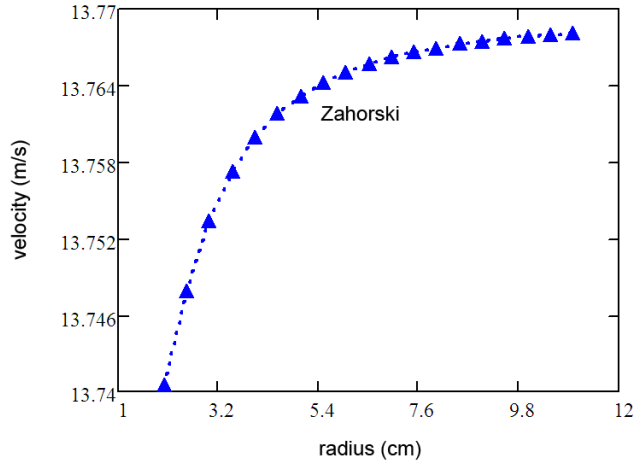
$$\psi(r, z) = \frac{1}{c_o} \left(1 + \frac{M_2 c}{2r^2} \right) z \quad (28)$$

This represents the equation of the front of acceleration wave for the Zahorski material.

4 NUMERICAL ANALYSIS

For the purposes of the numerical analysis, we assume a radial extension as a parameter that describes the deformation of $c = 0,3 \text{ cm}^2$, and rubber density of $\rho = 1190 \text{ kg/m}^3$.

a)



b)

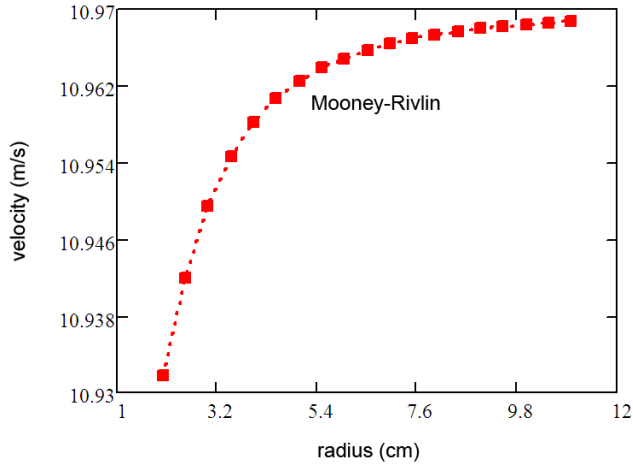


Fig. 2: Distribution of the velocity u_3 : a) Zahorski's material, b) Mooney – Rivlin material

The following constants C_1, C_2, C_3 were adopted from [5] for the rubber A (if $C_3 = 0$, we obtain the Mooney – Rivlin material):

$$C_{1A} = 6,278 \cdot 10^4 \text{ Pa}; \quad C_{2A} = 8,829 \cdot 10^3 \text{ Pa}; \quad C_{3A} = 6,867 \cdot 10^3 \text{ Pa};$$

The velocities of propagation u_3 are given by the formulae (17)₃, the radius of the cylinder ranges from 2 cm to 11 cm and its height was adopted as 10 cm. The diagrams that illustrate the relationship between the velocity and the radius for the Zahorski and Mooney – Rivlin materials are shown in Fig. 2. There are noticeable differences in the velocity of wave propagation between both materials.

Fig.3 illustrates the next location of the wave front for the velocity u_3 , the range $R \in \langle 2; 11 \rangle$ cm and deformation of $c = 0,3 \text{ cm}^2$ for two instants: $t = 1 \text{ s}$ and $t = 5 \text{ s}$. The front of the acceleration wave is given by the above equation (28). With $C_3 = 0$, we obtain the Mooney – Rivlin material.

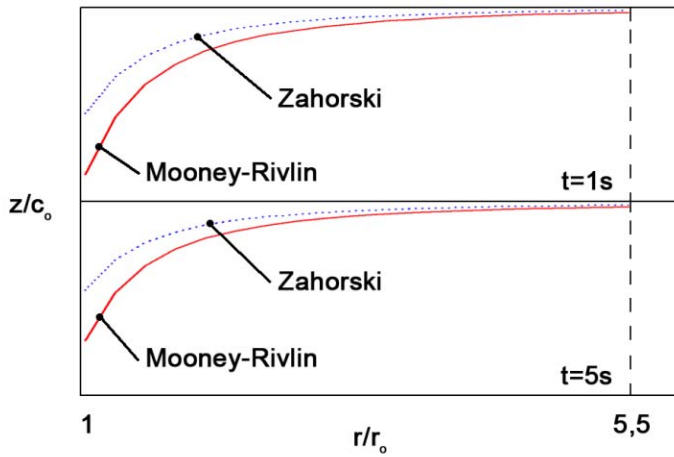


Fig. 3: The next location of the wave front for the velocity u_3 in the range $R \in \langle 2; 11 \rangle$ cm, $c = 0,3 \text{ cm}^2$ for two instants: $t = 1 \text{ s}$ and $t = 5 \text{ s}$.

5 CONCLUSIONS

The numerical analysis showed the differences in the distribution of the velocity u_3 between Zahorski and Mooney – Rivlin materials. This results from a non-linear dependence of the tensor I_1 on the invariant in the elastic Zahorski potential.

Also, the determination of the shape of the front of the propagating surface of discontinuities revealed specific quantitative differences, which were characterized by a greater curvature of the propagating wave front in the cylinder made of the Zahorski material compared to the Mooney – Rivlin material. The curvature of the wave front increases with time in both materials studied.

We found that, compared to the Mooney-Rivlin material, the Zahorski material adopted for the calculations shows significant differences in the velocity of propagation and curvature of the front of the acceleration wave.

The practical implications of using hyperelastic materials described with Zahorski potential concern rubber or rubber-like products which have been widely used in different branches of industry. While demonstrating the differences in the values of analytical solutions presented in this paper it is worth noting that the non-linear term present in the Zahorski potential (which is not

considered in the widely adopted Mooney-Rivlin potential) affects the analysis of wave phenomena that occur in rubber and rubber-like materials, which undoubtedly improves the quality of rubber mixtures used for specific purposes.

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